ELEC4410

Control System Design

Lecture 10: Elements of System Identification

School of Electrical Engineering and Computer Science
The University of Newcastle

Outline

- Introduction.

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- Types of Test Signals.

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Reference: Introduction to System Identification, Norton.
Dynamic System Identification, Goodwin & Payne.

Introduction

What is Identification?

Identification is the process of constructing a mathematical model of a (dynamical) system from observations of its inputs and outputs. Recall the definition of a dynamical system: the output at any instant depends on its history not just present input, i.e. it has some form of memory (storage).

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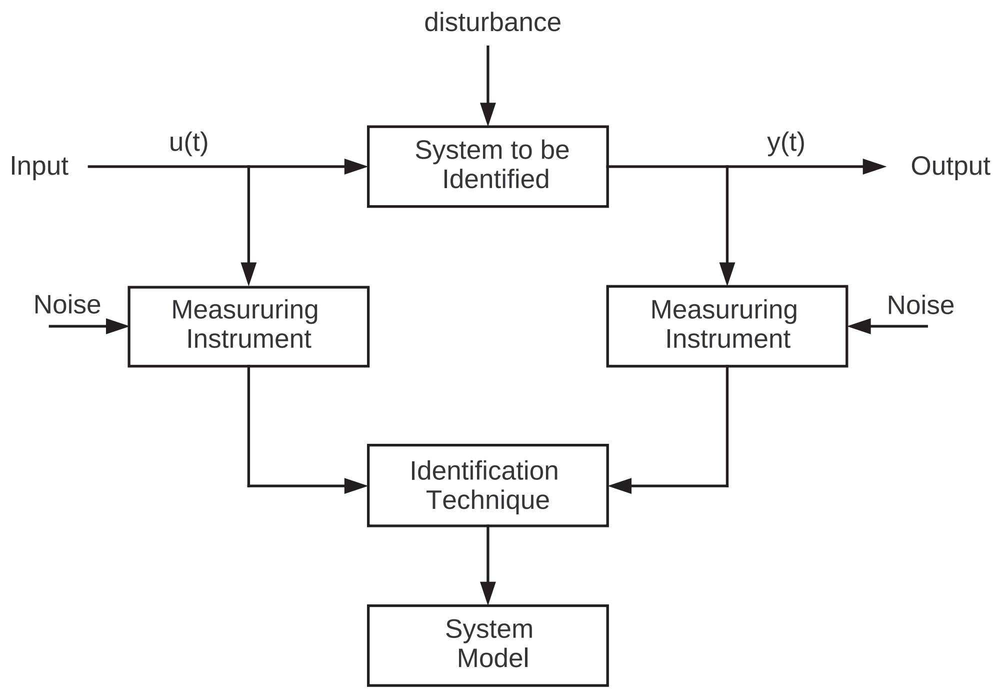

Classification of the System Identification Problem

Based on the degree of a priori knowledge of the system.

- **Black Box:** This means we know nothing about the basis properties of the system. Extremely difficult to solve. Usually some kind of assumptions have to be made before any meaningful solution can be attempted.

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- **Grey Box:** In this case, some basic characteristics of the system are known (ie. linearity, bandwidth, structure). However, order of the dynamic equation or values of the associated co-efficients may be unknown.

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- **Simulation:** Makes it possible to explore situations that would be either hazardous, difficult or prohibitively expensive. (i.e. evaluating the performance of different controllers on a model rather than on a nuclear reactor.)

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- In a **time invariant** system the input / output relationships do not vary with time. In a **time varying** system these relationships change with time.
- A **continuous time** model describes the system at any given instance of time. **Discrete time** models describe input and output relationships at distinct instances of time.

Types of Models

- **Nonparametric** models are characterised as curves or functions, not a set of parameters. e.g. nonparametric model consist of a time record of the impulse or step response in the time domain, or a frequency record of the transfer function in the frequency domain. e.g. Bode diagram. Essentially, an infinite number of measurements would be needed to represent the system. Practically, a “sufficiently” large number is required to “acceptably” represent the system. **Parametric** models concentrate all information in a model structure with a limited set of parameters. This makes the parametric model “economical” and powerful.

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A particular sort of model can be defined by a combination of these different types. e.g. One of the most common forms is the continuous time, linear, time invariant parametric model.

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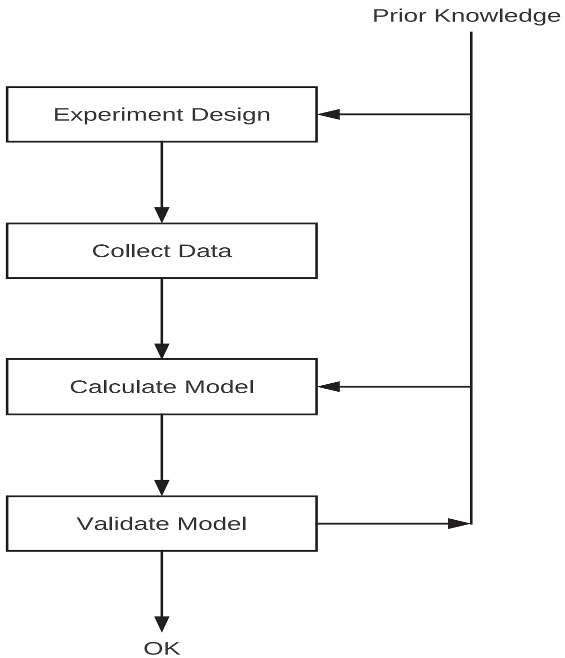

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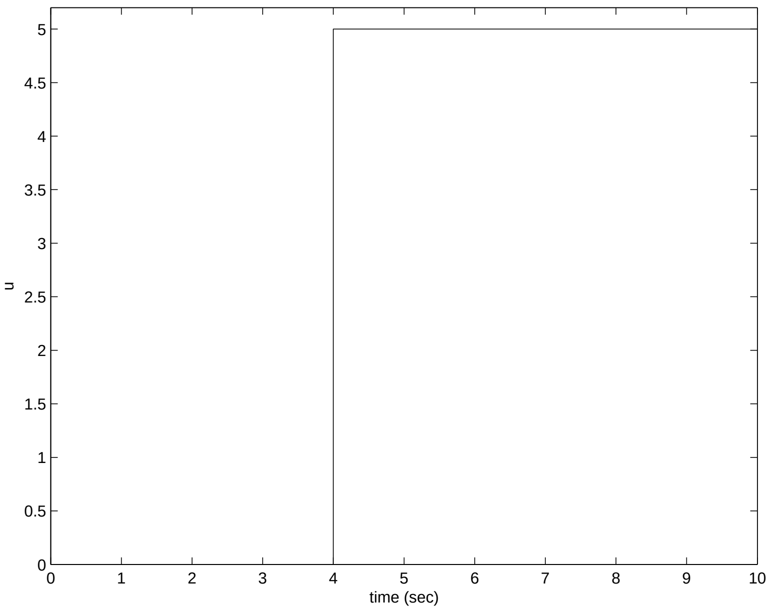
  - Pseudorandom binary sequence
  - Sinusoids
- The signals we consider in this course are step functions and sinusoids.

Types of Test Signals

Step Input Signal

$$u(t) = \begin{cases} 0 & t < 0, \\ u_f & t \geq 0. \end{cases}$$



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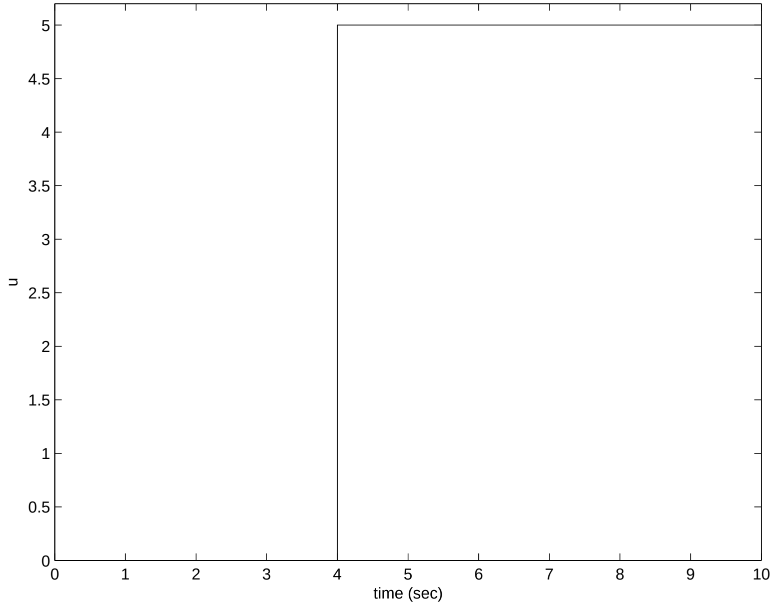

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- From the response of a process to a step input a number of practical parameters can be obtained, i.e. dead time, time constant, etc.

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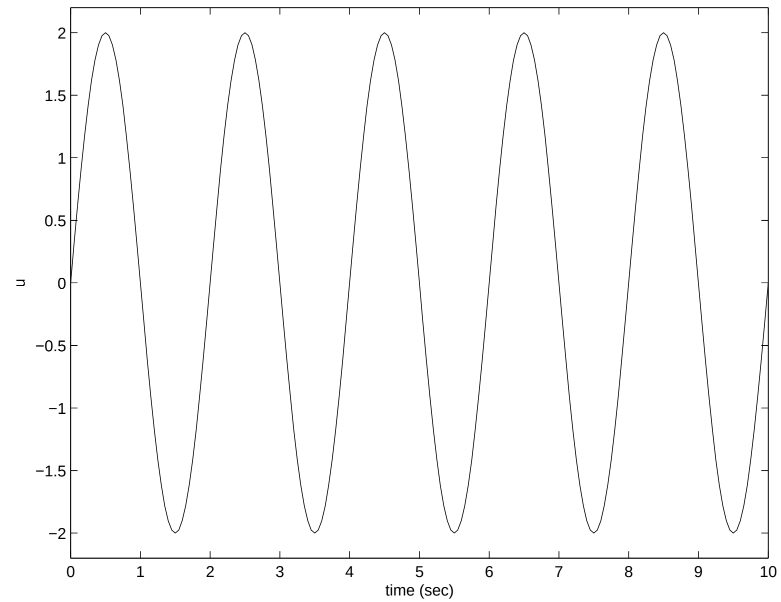
- User choices to be considered:
  - Amplitude (u_f)
  - Duration (T)

Types of Test Signals

Sinusoidal Signals

$$u(t) = a \sin(2\pi f t + \phi)$$



- Information in the frequency domain is most easily obtained by using sinusoidal or other periodic signals. Makes it suitable for finding continuous time models.

Types of Test Signals

Sinusoidal Signals

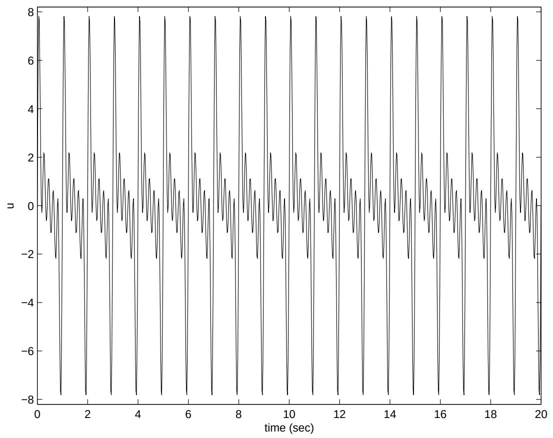
- Sinusoidal signals have many advantages:
  - limit the signal to frequencies of interest
  - duration of the test can be chosen quite arbitrarily
  - generation of the signal is quite straightforward
- The amplitude of the sinewave can be traded off for the duration of test, i.e. For a smaller amplitude you would perform the test for a longer time.
- A disadvantage is that one sinusoid only gives you one test frequency.
- To obtain an adequate model of a system you would need to perform a number of experiments.

Types of Test Signals

Multi-Sinusoidal Signals

$$u(t) = \sum_{k=1}^{m} a_k \sin(2\pi f_k t + \phi_k)$$

- As an example consider $m = 5$, $a_k = 2$, $\phi_k = 0 \;\; \forall k$ and $f = [1, \; 2, \; 3, \; 4, \; 5]$.

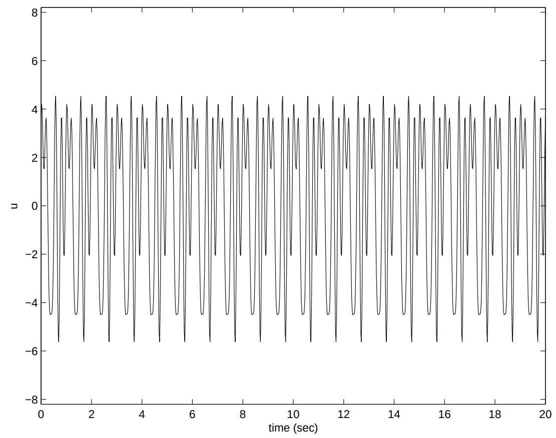


- Note the amplitude is large.

- To reduce this we can change the phase of each sine wave, i.e.

$$\phi_k = \phi_1 - \frac{k(k-1)\pi}{m}; \quad 2 \leq k \leq m$$

where ϕ_1 is chosen arbitrary.



Design of Identification Experiments

There are a number of factors to consider before performing an identification experiment.

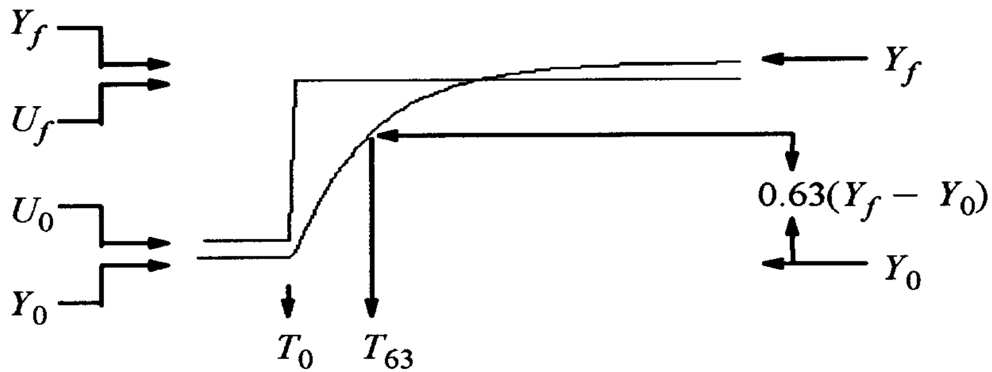
- What form of test signal should be used, i.e. step, sinusoidal? This depends on the quality of the model you require. In most cases, for PID control, step tests are adequate.
- What size should the amplitude be? There a quite a number of factors to consider here.
  - There may be constraints on how much variation can be tolerated in the input and/or output. (economic, safety, actuator limits, etc).
  - One reason for a large amplitude is that the effects of noise become less (signal to noise ratio is larger).
  - For systems with known nonlinearities it is best to keep the amplitude small as generally you are interested in a model around a particular operating point.

Design of Identification Experiments

- What sampling frequency should be used?
  - Typically, the sampling frequency should be chosen as 10 - 20× that of the test signal frequency. (5× should be considered as the absolute minimum).
  - For a step test, you want to capture at least 5 samples per the time constant of the process.
- What should the frequencies of my multi-sinusoidal test signal be? The frequencies of these test signals should be in the frequency region of interest and should be chosen as a multiple of the sampling frequency and of each other. This will minimise errors.
- When collecting data from a sinusoidal test one should wait until the transients have decayed significantly.

Estimating Transfer Functions from Step Responses

First Order Lag



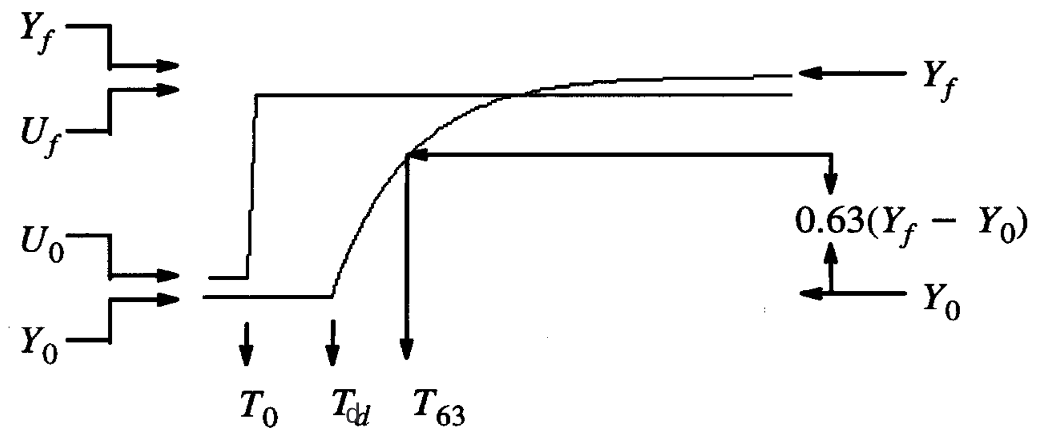
- Measure
  - U_0: Initial input level.
  - U_f: Final input level.
  - Y_0: Initial output level.
  - Y_f: Final output level.
  - T_0: Time of input step change.
  - T_{63}: Time for output to reach 63% of $(Y_f - Y_0)$.
- Calculate

$$\hat{K} = \frac{Y_f - Y_0}{U_f - U_0} \quad \text{and} \quad \hat{\tau} = T_{63} - T_0$$

$$\hat{G}(s) = \frac{\hat{K}}{\hat{\tau}s + 1}$$

Estimating Transfer Functions from Step Responses

Time Delayed First Order Lag



- Measure
  - U_0: Initial input level.
  - U_f: Final input level.
  - Y_0: Initial output level.
  - Y_f: Final output level.
  - T_0: Time of input step change.
  - T_d: Time at which system starts reacting to step.
  - T_{63}: Time for output to reach 63% of $(Y_f - Y_0)$.

(cont...)

Estimating Transfer Functions from Step Responses

Time Delayed First Order Lag (cont.)

- Calculate

$$\hat{K} = \frac{Y_f - Y_0}{U_f - U_0}$$

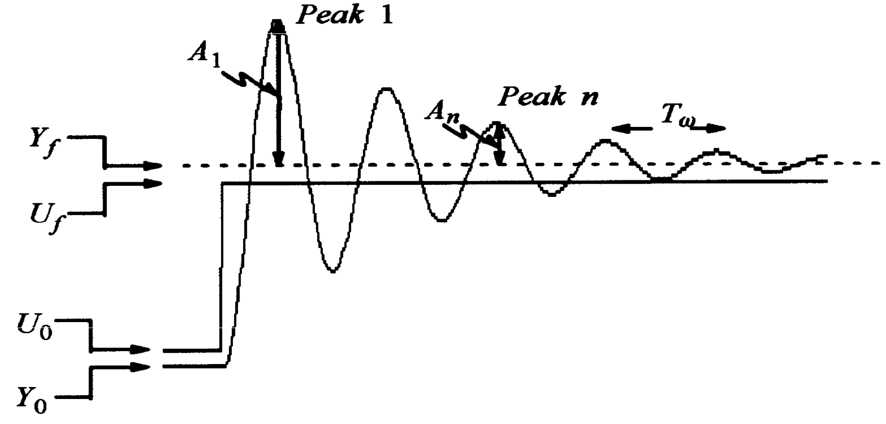
$$\hat{T}_d = T_d - T_0$$

$$\hat{\tau} = T_{63} - T_d$$

$$\hat{G}(s) = \frac{\hat{K}e^{-s\hat{T}_d}}{\hat{\tau}s + 1}$$

Estimating Transfer Functions from Step Responses

Second Order Resonant System



- Measure
  - U_0: Initial input level.
  - U_f: Final input level.
  - Y_0: Initial output level.
  - Y_f: Final output level.
  - Peak 1: An arbitrary peak.
  - Peak n: Peak n counting from Peak 1: (n = 3 in Figure).
  - A_1: Amplitude from Y_f to Peak 1.
  - A_n: Amplitude from Y_f to Peak n
  - T_w: Time between two successive peaks. (cont...)

Estimating Transfer Functions from Step Responses

Second Order Resonant System (cont.)

- Calculate

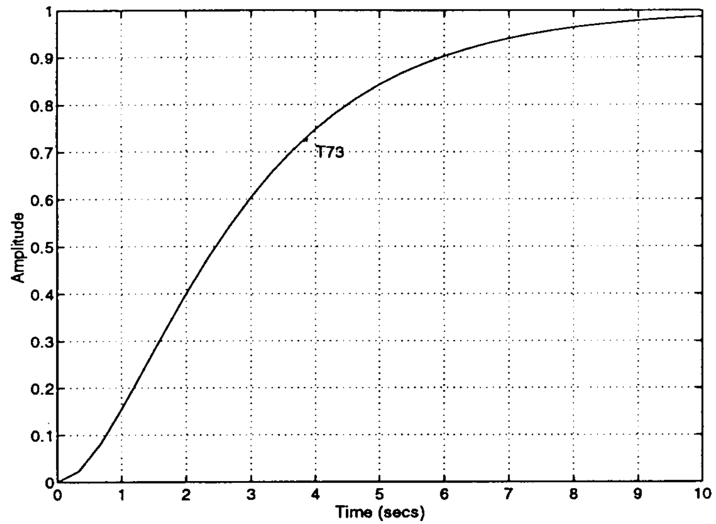
$$\hat{K} = \frac{Y_f - Y_0}{U_f - U_0}, \qquad d_r = \left(\frac{A_n}{A_1}\right)^{\frac{1}{n-1}},$$

$$\hat{\zeta} = \frac{\ln\left(\frac{1}{d_r}\right)}{\sqrt{4\pi^2 + \left(\ln\left(\frac{1}{d_r}\right)\right)^2}}, \qquad \hat{T}_n = \frac{T_w\sqrt{1-\hat{\zeta}^2}}{2\pi}.$$

$$\hat{G}(s) = \frac{\hat{K}}{\hat{T}_n^2 s^2 + 2\hat{\zeta}\hat{T}_n s + 1}$$

Estimating Transfer Functions from Step Responses

Second Order Overdamped System



- Measure
  - U_0: Initial input level.
  - U_f: Final input level.
  - Y_0: Initial output level.
  - Y_f: Final output level.
  - T_0: Time of input step change.
- T_{73}: Time of output to reach 73% of $(Y_f - Y_0)$.
- $\acute{Y}$: Value of output at time:

$$\left(T_0 + \frac{T_{73} - T_0}{2.6}\right).$$

Estimating Transfer Functions from Step Responses

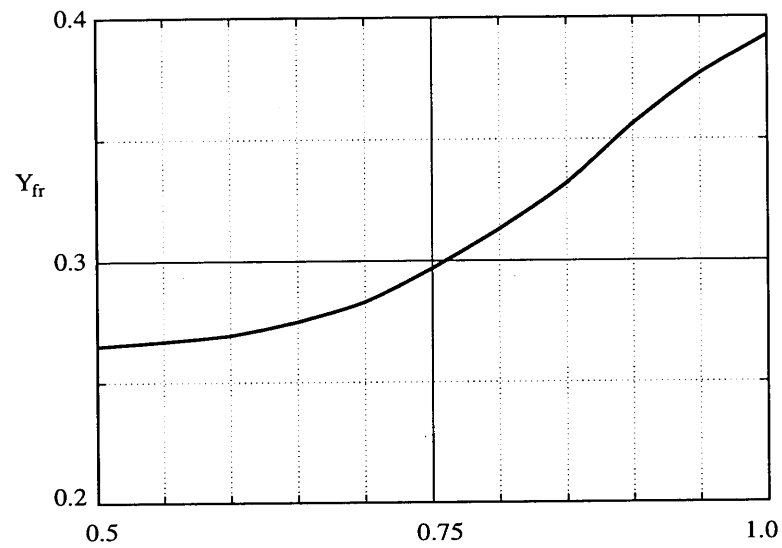
Second Order Overdamped System (cont.)

- Calculate

$$\hat{K} = \frac{Y_f - Y_0}{U_f - U_0}, \quad \hat{\tau}_{TOT} = \frac{T_{73} - T_0}{1.3}$$

and $Y_{fr} = \dfrac{\acute{Y} - Y_0}{Y_f - Y_0}$

- Find $\hat{\tau}_{rat}$ from Y_{fr} using the supplied graph and compute



$$\hat{\tau}_1 = \hat{\tau}_{rat}\hat{\tau}_{TOT}$$
$$\hat{\tau}_2 = \hat{\tau}_{TOT} - \hat{\tau}_1$$

- The estimated model is:

$$\hat{G}(s) = \frac{\hat{K}}{(\hat{\tau}_1 s + 1)(\hat{\tau}_1 s + 1)}$$

NOTE: If Y_{fr} is greater than 0.39 or less than 0.26, the response is either underdamped second order or higher order.

Estimating Transfer Functions from Step Responses

Integrating System

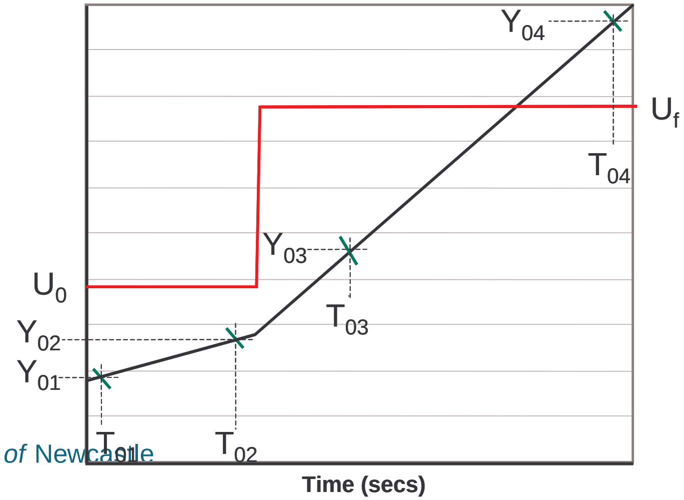
- Measure
  - U_0: Initial input level.
  - U_f: Final input level.
  - Y_{01}: Output level at time T_{01} (before step).
  - Y_{02}: Output level at time T_{02} (before step).
  - Y_{03}: Output level at time T_{03} (after step).
  - Y_{04}: Output level at time T_{04} (after step).
- Calculate

$$m_1 = \frac{Y_{02} - Y_{01}}{T_{02} - T_{01}}$$

$$m_2 = \frac{Y_{04} - Y_{03}}{T_{04} - T_{03}}$$

$$\hat{K} = \frac{m_2 - m_1}{U_f - U_0}$$

$$\hat{G}(s) = \frac{\hat{K}}{s}$$



Frequency Analysis

- Recall, for the system

that

$$Y(s) = G(s)U(s)$$

- If we apply a sinewave to the input,

$$u(t) = a\sin(\omega t)$$

$$\text{(remember)}\quad \omega = 2\pi f$$

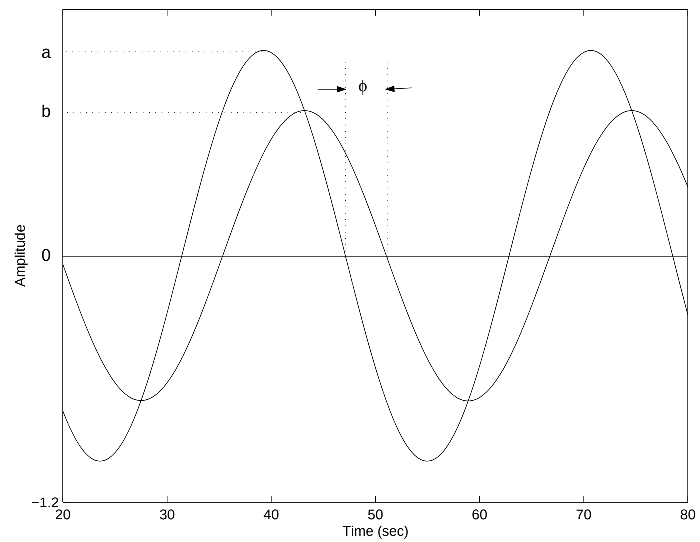
and G is stable and in steady state, then

$$y(t) = b\sin(\omega t + \phi)$$

$$\text{where}\quad b = a\,|G(j\omega)|$$

$$\phi = \arg\left(G(j\omega)\right)$$

- Hence, by measuring b and ϕ, we can obtain an estimate of G at frequency ω.



Frequency Analysis

- By repeating this for a number of frequencies, one can obtain a reasonable graphical representation of the process, i.e. the Bode Diagram, a non-parametric model.
- In practice, this type of measurement is sensitive to noise,

$$Y(s) = G(s)U(s) + V(s)$$

  where $V(s)$ is a representation of the noise appearing at the output of the system.
- Then

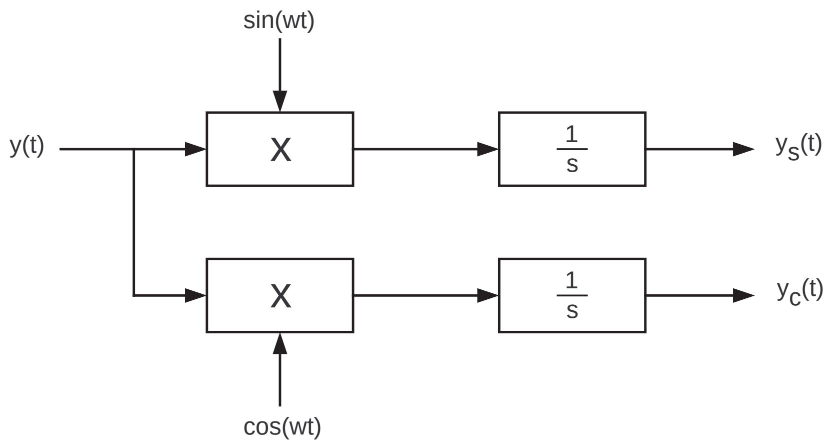
$$y(t) = b\sin(\omega t + \phi) + e(t)$$

  which introduces errors into the measurement of $y(t)$ and hence the parameters b and ϕ.

Frequency Analysis

An Improved Frequency Analysis Method

To improve estimation of the frequency response, correlate the output $y(t)$ with $\sin$ and $\cos$ at the desired frequency.



Frequency Analysis

$$y_s(t) = \int_0^T y(t)\sin(\omega t)\mathrm{d}t$$

$$= \int_0^T b\sin(\omega t + \phi)\sin(\omega t)\mathrm{d}t + \int_0^T e(t)\sin(\omega t)\mathrm{d}t$$

(using the identity: $2\sin(A)\sin(B) = \cos(A - B) - \cos(A + B)$)

$$= \int_0^T \frac{b}{2}\cos(\phi)dt - \int_0^T \frac{b}{2}\cos(2\omega t + \phi)dt + \int_0^T e(t)\sin(\omega t)\mathrm{d}t$$

$$= \frac{bT}{2}\cos(\phi) + \int_0^T e(t)\sin(\omega t)\mathrm{d}t$$

Note: If integration time (T) is a multiple of the sinusoidal period, say $\frac{k2\pi}{\omega}$ then the second term in the fourth line above $= 0$.

Frequency Analysis

Similarly,

$$y_c(t) = \frac{bT}{2}\sin(\phi) + \int_0^T e(t)\cos(\omega t)dt$$

Let us first consider $e(t) = 0$. Then

$$y_s(t) = \frac{bT}{2}\cos(\phi)$$
$$y_c(t) = \frac{bT}{2}\sin(\phi)$$

Recall,

$$b = a\,|G(j\omega)|$$
$$\phi = \arg\left(G(j\omega)\right)$$

Frequency Analysis

Then

$$y_s(t) = \frac{aT}{2} |G(j\omega)| \cos(\arg(G(j\omega)))$$

and $$y_c(t) = \frac{aT}{2} |G(j\omega)| \sin(\arg(G(j\omega)))\,.$$

Now $$G(j\omega) = |G(j\omega)|\, e^{j(\arg(G(j\omega)))}$$

$$= |G(j\omega)| \left[\cos(\arg(G(j\omega))) + j\sin(\arg(G(j\omega)))\right]$$

We then have

$$y_s(t) = \frac{aT}{2} \Re\{G(j\omega)\}$$

$$y_c(t) = \frac{aT}{2} \Im\{G(j\omega)\}$$

Frequency Analysis

- Therefore we can calculate real and imaginary parts $G(j\omega)$, hence construct a Bode Diagram.
- Now if $e(t) \neq 0$, then we still get errors in the estimate!
- However, as $T \uparrow$ the error decreases. (In the case of i.i.d. noise). Due to the fact that it is not correlated with the $\sin$ and $\cos$ terms.

Frequency Analysis

Can also use discrete Fourier Transforms.

$$U_N(\omega) = \sum_{t=1}^{N} u(t)e^{-j\omega t}$$

$$Y_N(\omega) = \sum_{t=1}^{N} y(t)e^{-j\omega t}$$

Then $$G(j\omega) = \frac{Y_N(\omega)}{U_N(\omega)}$$

Notes:

- Works best for periodic signals.
- For best results, N should be an integer multiple of the periodic signal.
- Scaling not really necessary as we are mainly concerned with the ratio.

Least Squares Model Fitting

The Least Squares method estimates the coefficients for a given model by minimising the sum of squared errors between the observations and the model output.

- large errors are heavily punished, an error twice as large is four times worse.
- uses quite simple matrix algebra
- estimates are computed as a solution to a set of linear equations.

The model is required to

- relate observed variable $y(t)$ (regressand), to p explanatory variables $u_{1t} \ldots u_{pt}$ (regressors), all of which are observed.
- have one unknown coefficient θ per explanatory variable.

Least Squares Model Fitting

At one time instance, t

$$\underline{u}_t = \begin{bmatrix} u_{1t} & u_{2t} & \ldots & u_{pt} \end{bmatrix}^T , \qquad \underline{\theta} = \begin{bmatrix} \theta_1 & \theta_2 & \ldots & \theta_p \end{bmatrix}^T$$

the model is then

$$y_t = \underline{u}_t^T \underline{\theta} + e_t \ , \quad t = 1, 2, 3, \ \ldots \ N$$

where e_t is the observation error.

The aim is to find the value $\underline{\hat{\theta}}$ of $\underline{\theta}$ which minimises the cost function,

$$\hat{\theta} = \arg\min_{\theta} V(\theta)$$

$$\text{where} \qquad V(\theta) \triangleq \sum_{t=1}^{N} e_t^2 = \sum_{t=1}^{N} \left(y_t - \underline{u}_t^T \underline{\theta} \right)^2$$

Least Squares Model Fitting

For N samples, $\underline{y}$ is a N length vector, the $\underline{u}_t$ vectors form into an $N \times P$ matrix U and a N length vector $\underline{e}$ is formed by the errors. Then,

$$\underline{y} = U\underline{\theta} + \underline{e}$$

$$\text{and} \quad V(\theta) = \underline{e}^T \underline{e}$$

$$= \left(\underline{y}^T - \underline{\theta}^T U^T\right)\left(\underline{y} - U\underline{\theta}\right).$$

The value $\underline{\hat{\theta}}$ that minimises V makes the gradient of V with respect to $\underline{\theta}$ zero, i.e

$$\frac{\partial V}{\partial \underline{\theta}} = \left[\frac{\partial V}{\partial \theta_1} \; \frac{\partial V}{\partial \theta_2} \; \cdots \; \frac{\partial V}{\partial \theta_p}\right]^T = \mathbf{0}$$

Now

$$V(\theta) = \underline{y}^T \underline{y} - \underline{\theta}^T U^T \underline{y} - \underline{y}^T U\underline{\theta} + \underline{\theta}^T U^T U\underline{\theta}$$

Least Squares Model Fitting

Using standard results for vector and matrix differentiation,

$$\frac{\partial(\underline{a}^T\underline{\psi})}{\partial\underline{\psi}} = \underline{a}$$

$$\frac{\partial(\underline{\psi}^T A\underline{\psi})}{\partial\underline{\psi}} = (A + A^T)\underline{\psi}.$$

Then

$$\frac{\partial V}{\partial\underline{\theta}} = 2U^T\underline{y} + 2U^T U\underline{\theta} = \mathbf{0}.$$

Thus the $\underline{\theta}$ that makes the gradient of $V(\theta) = \mathbf{0}$ is given by,

$$\underline{\hat{\theta}} = \left[U^T U\right]^{-1} U^T\underline{y}$$

Least Squares Model Fitting

Notes:

- The model must be linear in the unknown coefficients
- It need not be linear in the regressors
- Be careful of an ill-conditioned normal matrix!

Least Squares Model Fitting

Example: Consider a temperature measuring device with a voltage output, u. It is known that the temperature, y, is a function of the output voltage. The model is given by,

$$y(u) = x_1 + x_2 u + x_3 \frac{u^2}{2}.$$

The observations are:

$$\underline{u} = \begin{bmatrix} 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1 \end{bmatrix} \quad \text{(volts)}$$

$$\underline{y} = \begin{bmatrix} 3 & 59 & 98 & 151 & 218 & 264 \end{bmatrix}^T \quad {}^\circ\text{(C)}$$

and the parameter vector is

$$\underline{\theta} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$$

Least Squares Model Fitting

Example: cont.

1. Now we form,

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.2 & 0.02 \\ 1 & 0.4 & 0.08 \\ 1 & 0.6 & 0.18 \\ 1 & 0.8 & 0.32 \\ 1 & 1 & 0.5 \end{bmatrix}$$

$$U^T \underline{y} = \begin{bmatrix} 793 \\ 580 \\ 237.96 \end{bmatrix}$$

$$U^T U = \begin{bmatrix} 6 & 3 & 1.1 \\ 3 & 2.2 & 0.9 \\ 1.1 & 0.9 & 0.3916 \end{bmatrix}$$

2. We can then solve for an estimate of the parameters using

$$\underline{\hat{\theta}} = \left[U^T U\right]^{-1} U^T \underline{y}$$
$$= \begin{bmatrix} 4.79 & 234 & 55.4 \end{bmatrix}^T$$

Frequency Domain Parametric Models

Consider the model,

$$G(s) = \frac{B(s)}{A(s)}$$

$$\text{where} \quad B(s) = b_0 + b_1 s + \ldots + b_{n-1} s^{n-1}$$

$$\text{and} \quad A(s) = 1 + a_1 s + \ldots + a_n s^n.$$

We perform an experiment which consists of applying sinewaves of frequency $\omega_1, \quad \omega_2, \quad \ldots \quad \omega_N$ to the system. The DFT can be used to obtain a nonparametric model of the system

$$\hat{G}(j\omega).$$

Note that ˆ is used here as there will be unavoidable errors in the measurement.

Frequency Domain Parametric Models

Now

$$A(j\omega)G(j\omega) = B(j\omega)$$

then $$V(\theta) = \sum_{i=1}^{N} e_i^* e_i$$

where $$e_i = A(j\omega_i)\hat{G}(j\omega_i) - B(j\omega_i)$$

here the i subscript represents the ith frequency of the test signal. We temporarily drop the i subscript for clarity,

$$\begin{aligned} e &= [1 + a_1 j\omega + \ldots + a_n(j\omega)^n]\,\hat{G}(j\omega) - \left[b_0 + b_1 j\omega + \ldots + b_{n-1}(j\omega)^{n-1}\right] \\ &= \hat{G}(j\omega) - \left[-j\omega\hat{G}(j\omega), \quad \ldots, \quad -(j\omega)^n\hat{G}(j\omega), \quad 1, \quad j\omega, \quad \ldots, \quad (j\omega)^{n-1}\right]\underline{\theta} \end{aligned}$$

where $$\underline{\theta} = \left[a_1 \quad \ldots \quad a_n \quad b_o \quad \ldots \quad b_{n-1}\right]^T$$

Frequency Domain Parametric Models

The cost function $V(\theta)$ can now be expressed as

$$V(\theta) = (\underline{Y} - U\underline{\theta})^*(\underline{Y} - U\underline{\theta})$$

where

$$\underline{\theta} = \begin{bmatrix} a_1 & a_2 & \ldots & a_n & b_o & \ldots & b_{n-1} \end{bmatrix}^T$$

$$\underline{Y} = \begin{bmatrix} \hat{G}(j\omega_1) & \hat{G}(j\omega_1) & \ldots & \hat{G}(j\omega_N) \end{bmatrix}^T$$

$$U = \begin{bmatrix} -j\omega_1\hat{G}(j\omega_1) & \ldots & -(j\omega_1)^n\hat{G}(j\omega_1) & 1 & j\omega_1 & \ldots & (j\omega_1)^{n-1} \\ \vdots & & & & & & \vdots \\ -j\omega_N\hat{G}(j\omega_N) & \ldots & -(j\omega_N)^n\hat{G}(j\omega_N) & 1 & j\omega_N & \ldots & (j\omega_N)^{n-1} \end{bmatrix}$$

Frequency Domain Parametric Models

Using the same procedure as before to find the minimum gradient of the cost function,

$$\frac{\partial V}{\partial \underline{\theta}} = -U^* \left(\underline{Y} - U\underline{\theta}\right) - U^T \left(\overline{\underline{Y} - U\underline{\theta}}\right)$$

$$= -\left(U^* \underline{Y} + U^T \overline{\underline{Y}}\right) + \left(U^* U + U^T \overline{U}\right) \underline{\theta}$$

set $$\frac{\partial V}{\partial \underline{\theta}} = \mathbf{0}$$

then $$\underline{\hat{\theta}} = \left(U^* U + U^T \overline{U}\right)^{-1} \left(U^* \underline{Y} + U^T \overline{\underline{Y}}\right)$$

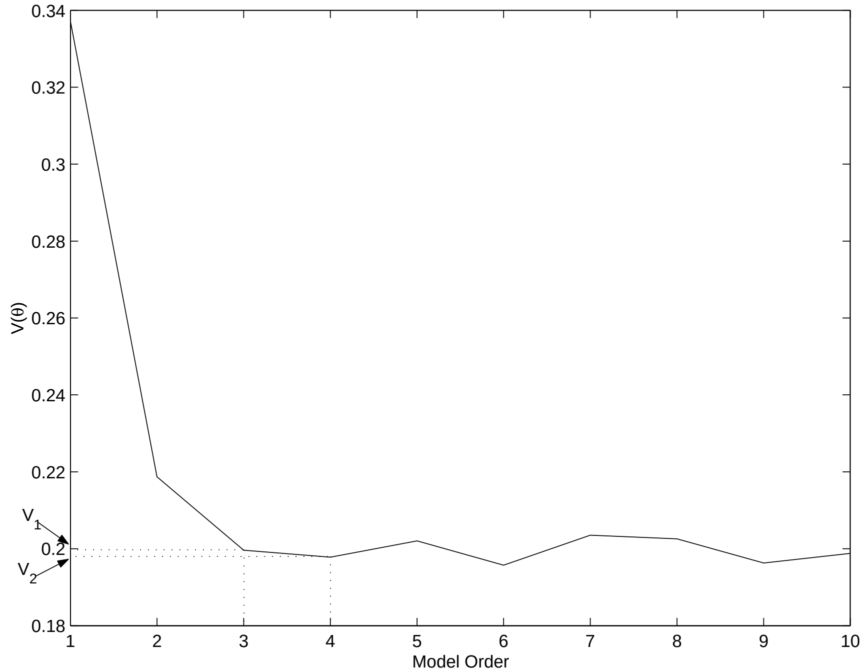
$$= \left(U^* U + \overline{U^* U}\right)^{-1} \left(U^* \underline{Y} + \overline{U^* \underline{Y}}\right)$$

$$= \left(\Re\left\{U^* U\right\}\right)^{-1} \left(\Re\left\{U^* \underline{Y}\right\}\right)$$

Model Order Determination

A number of possibilities, two of these are:

- Use the Bode diagram to identify poles / zeros
- Calculate the L.S. model for an increasing number of parameters, and evaluate the cost function $V(\hat{\theta})$ for each model. Look for small ΔV. ($\Delta V = |V_1 - V_2|$).



Model Validation

- Is a given model appropriate? i.e. does it meet the expectation you have of it representing the system?
- Most systematic methods of validation are based on statistics.
- An ad-hoc (simple method) is to visually approve the model by observing the output of the model and the true system for the same input signal.

Model Validation

Systematic Methods

- If the model is a true representation of the system and the "disturbance" is assumed to be independent white noise, then the residuals,

$$\underline{e} = \underline{y} - u\underline{\hat{\theta}}$$

  should also be independent white noise.

- Notes:
  - Never use the same data for validation that was used for estimation.
  - Least Squares estimation makes e uncorrelated with the regressors on which the data the estimation is performed.

Model Validation

Changes of sign method

- Let

$$\delta_k = \begin{cases} 1 & : \ e(k)e(k+1) < 0 \ \text{(change of sign)} \\ 0 & : \ e(k)e(k+1) > 0 \ \text{(no change of sign)} \end{cases}$$

  and

$$X_n = \sum_{k=1}^{N-1} \delta_k$$

- For white independent residuals,

$$mean(X_n) = \frac{N}{2}$$
$$variance(X_n) = \frac{N}{4}$$

Model Validation

Correlation Between Residuals and Past Inputs

$$\hat{R}_{eu}^{N}(\tau) = \frac{1}{N}\sum_{t=1}^{N} e(t)u(t-\tau)$$

- If the correlation is "high", there may be some of the input contained in the residuals e and hence not taken into account in the model.
- Ideally we want no correlation between input and residuals.

$$\hat{R}_{eu}^{N}(\tau) \leq \alpha \sqrt{\frac{P_1}{N}}$$

where α represents a confidence value and

$$P_1 = \sum_{k=-\infty}^{\infty} R_e(k)R_u(k)$$

$R_e(k)$ and $R_u(k)$ are the autocorrelation of $e(t)$ and $u(t)$ respectively.

Model Validation

Correlation Between Residuals and Past Inputs (cont.)

- Typically plot $\hat{R}_{eu}^{N}(\tau)$ and the lines $\pm 3\sqrt{\frac{P_1}{N}}$.
- If $\hat{R}_{eu}^{N}(\tau)$ goes outside of this then most probably due to $e(t)$ and $u(t-\tau)$ being dependant.

Model Validation

Correlation between residuals

$$\hat{R}(\tau) = \frac{1}{N} \sum_{t=1}^{N} e(t)e(t+\tau)$$

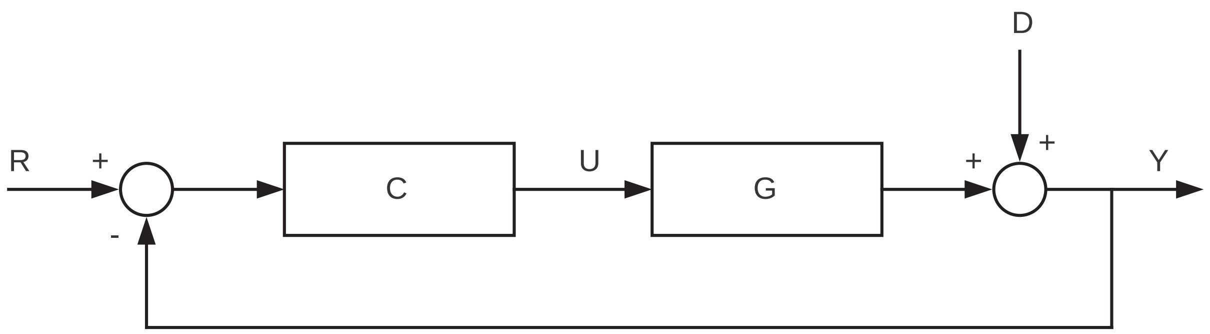
- Should be white.
- Plot $\hat{R}(\tau)$ against τ.

Other measures used in validation:

- Maximum Error: Largest Residual `{max(e)}`.
- Mean Square Error: $MSE = \frac{1}{N} \sum_{t=1}^{N} e(t)^2$.
- Root Mean Square Error: $RMSE = \sqrt{MSE}$.

Identification in Closed Loop

- Until now what has been considered is open loop identification. If we use the same principles in closed loop, we must be careful.
- Two commonly used types of identification are:
  - Direct: measure U and Y then identify G.
  - Indirect: measure R and Y then identify G using knowledge of C.



Identification in Closed Loop

For example lets consider direct identification:

- Now we want to estimate G.

$$\hat{G} = \frac{Y}{U} = \frac{D + GU}{U} = \frac{D}{U} + G.$$

- Now let U_R denote the controller output signal due to the reference R, then

$$U_R = C(R - GU_R)$$

$$U_R = \frac{CR}{1 + GC}$$

- Let U_D denote the controller output due to the disturbance D,

$$U_D = -C(D + GU_D)$$

$$U_D = \frac{-CD}{1 + GC}.$$

Identification in Closed Loop

- We are assuming a linear system, then by superposition

$$U = U_R + U_D = \frac{CR - CD}{1 + GC}$$

- Then

$$\begin{aligned}\hat{G} &= \frac{D(1 + GC)}{CR - CD} + G \\ &= \frac{D + GCR}{CR - CD} \\ &= \frac{1}{R - D}\left(\frac{D}{C} + GR\right)\end{aligned}$$

Identification in Closed Loop

- Defining the noise to signal ratio as $\alpha = \frac{D}{R}$,

$$\hat{G} = \frac{\frac{1}{R}}{1 - \frac{D}{R}} \left(\frac{D}{C} + GR \right)$$

$$= \left(\frac{\alpha}{1-\alpha} \right) \left(\frac{1}{C} \right) + \left(\frac{1}{1-\alpha} \right) G.$$

- When $D = 0$, $\alpha = 0$ therefore $\hat{G} = G$.
- When $D >> R$, α is large and $\hat{G} \Rightarrow \frac{-1}{C}$.
- Therefore in closed loop identification one needs to ensure the reference signal is larger than the disturbance, ($R > D$).