ELEC4410

Control Systems Design

Lecture 13: Stability

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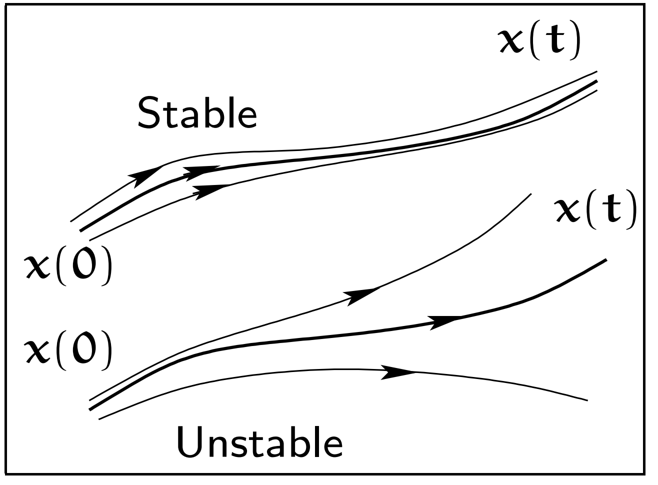
Outline

- Input-Output Stability of LTI systems
- Internal Stability of LTI systems
- Stability of Discrete-time systems

Stability

The stability of a system can be thought as a **continuity** in its dynamic behaviour. If a **small** perturbation arises in the system inputs or initial conditions, a stable system will present **small** modifications in its perturbed response.

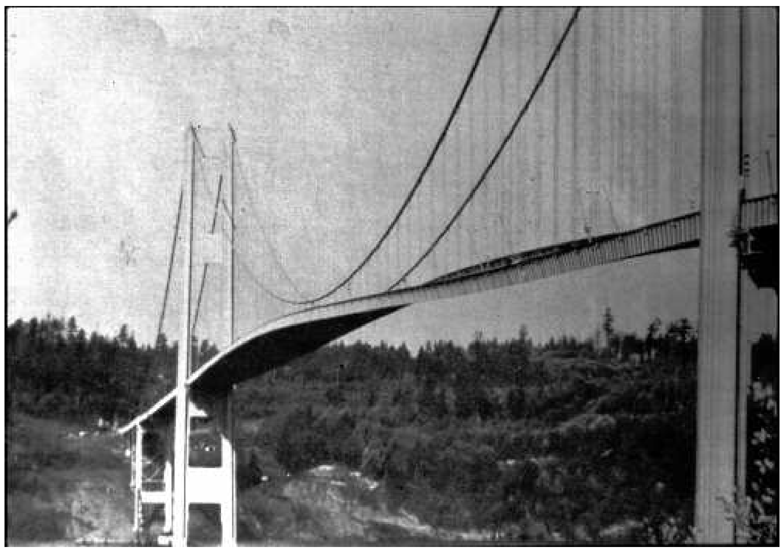
In an unstable system, any perturbation, no matter how small, will make states or outputs grow unbounded or until the system disintegrates or saturates.



Stability is a basic requirement of dynamic systems that perform operations or process signals; the first objective in control design.

The Tacoma Narrows Bridge Failure

On November 7, 1940, at approximately 11:00 AM, the first Tacoma Narrows suspension bridge collapsed due to wind-induced vibrations. Situated on the Tacoma Narrows in Puget Sound, near the city of Tacoma, Washington, the bridge had only been open for traffic a few months.



The Tacoma Narrows Bridge Failure is a formidable example of a system that was built with a structural instability.

Input Output Stability of LTI Systems

Recall that the response of a LTI system is composed of

$$\boxed{\text{response to initial conditions}} + \boxed{\text{response to inputs}}$$

The concept of **Input-Output Stability** refers to stability of the **response to inputs** only, assuming zero initial conditions.

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Theorem (BIBO Stability and Impulse Response). A SISO system is BIBO stable if and only if its impulse response $g(t)$ is **absolutely integrable** in the interval $[0, \infty)$. That is, if

$$\int_0^\infty |g(\tau)| d\tau \leq M$$

for some finite constant $M \geq 0$.

Input Output Stability of LTI Systems

Theorem (BIBO Stability and Steady State Response). If a system with transfer (matrix) function $G(s)$ is BIBO stable, then as $t \to \infty$

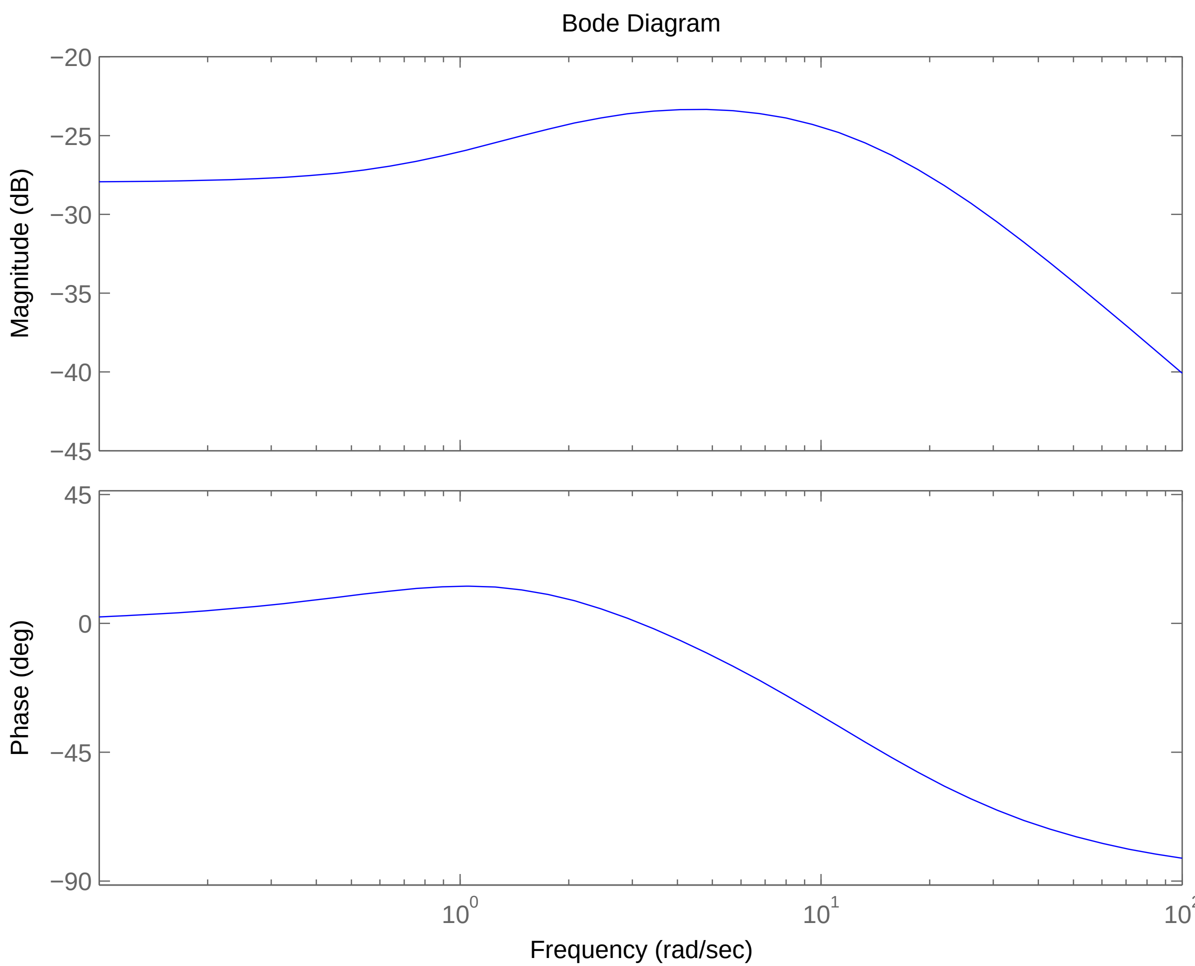
1. The output excited by $u(t) = a$, for $t \geq 0$, approaches
$$G(0)a.$$
2. The output excited by $u(t) = \sin(\omega_0 t)$, for $t \geq 0$, approaches
$$|G(j\omega_0)| \sin(\omega_0 t + \measuredangle G(j\omega_0)),$$

This is a basic result: specifies the response of a BIBO system to constant and sinusoidal signals once the transients have extinguished. Filtering of signals is based essentially on this theorem.

Input Output Stability of LTI Systems

Example. Take the transfer function $G(s) = \frac{s+1}{s^2+15s+26}$. It has poles at $s = -13$ and $s = -2$, so it is BIBO stable.

The Bode diagram plots the gains and phases of $G(j\omega)$.



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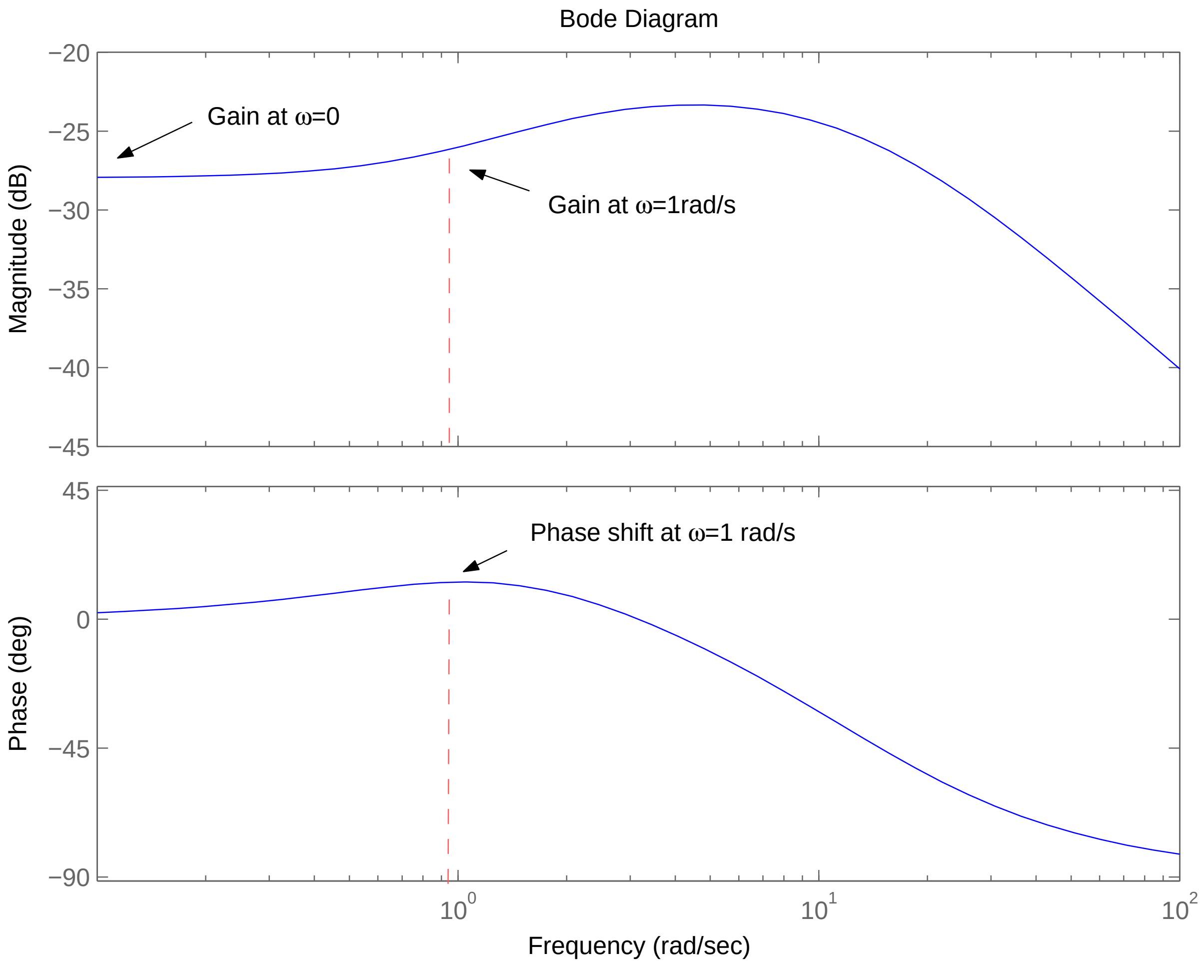

Input Output Stability of LTI Systems

Theorem (BIBO Stability). A LTI system with proper rational transfer matrix $\mathbf{G}(s) = [\mathbf{G}_{ij}(s)]$ is BIBO stable if and only if every pole of every entry $\mathbf{G}_{ij}(s)$ of $\mathbf{G}(s)$ has negative real part.

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BIBO stability of state equations. When the system is represented by state equations

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

the BIBO stability will depend on the eigenvalues of the matrix $\mathbf{A}$, since every pole of $\mathbf{G}(s)$ is an eigenvalue of $\mathbf{A}$.

Input Output Stability of LTI Systems

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BIBO stability of state equations. When the system is represented by state equations

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

the BIBO stability will depend on the eigenvalues of the matrix A, since every pole of $\mathbf{G}(s)$ is an eigenvalue of A. Indeed

$$\mathbf{G}(s) = C(sI - A)^{-1}B + D = \frac{C\,\mathrm{adj}(sI - A)B}{\det(sI - A)} + D,$$

thus if *all* eigenvalues of A have negative real part, all the poles of $\mathbf{G}(s)$ will have negative real part, and the system will be BIBO stable.

Input Output Stability of LTI Systems

Note that **not every eigenvalue of A is a pole of $G(s)$**, since there may be pole-zero cancellations while computing $G(s)$. Thus, a state equation may be BIBO stable even when some eigenvalues of A do not have negative real part.

Input Output Stability of LTI Systems

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Example. Although the system

$$\dot{x}(t) = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -2 & 3 \end{bmatrix} x(t) - 2u(t)$$

has one eigenvalue with positive real part $\lambda = 1$, it is BIBO stable, since its transfer function

$$G(s) = C(sI - A)^{-1}B + D = \frac{2(1-s)}{(s+1)}$$

has a single pole at $s = -1$. $\square$

Outline

- Input-Output Stability of LTI systems
- Internal Stability of LTI systems
- Stability of Discrete-time systems

Internal Stability of LTI Systems

The concept of **Internal Stability** refers to stability of the system **response to initial conditions** only, assuming zero inputs.

In other words, we now study the stability of the response of the state equation

$$\dot{x}(t) = Ax(t), \qquad \text{with } x(0) = x_0.$$

Because the solution of this equation is given by

$$x(t) = e^{At}x_0,$$

stability is determined by the eigenvalues of A, as we discussed when we studied the state response to initial conditions. Loosely speaking, if the eigenvalues of A have all negative real part, the system response will decay to 0 as $t \to \infty$.

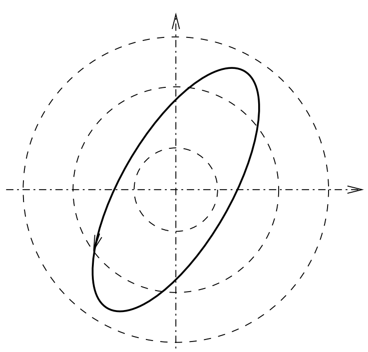
We make the definition of internal stability more precise.

Internal Stability of LTI Systems

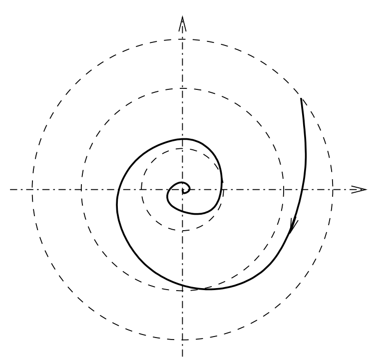
Lyapunov Stability. The system $\dot{x}(t) = Ax(t)$ is Lyapunov stable, or marginally stable, or simply stable, if *every* finite initial state x_0 excites a bounded response $x(t)$.

Asymptotic Stability. The system $\dot{x}(t) = Ax(t)$ is asymptotically stable if *every* finite initial state x_0 excites a bounded response $x(t)$ that approaches 0 as $t \to \infty$.

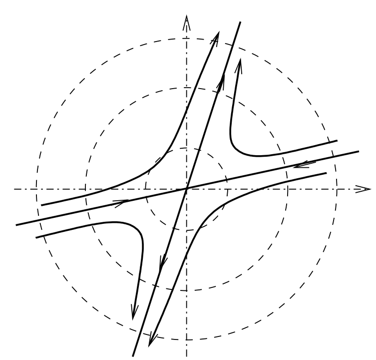
Instability. The system $\dot{x}(t) = Ax(t)$ is unstable if it is not stable.



Lyapunov Stability



Asymptotic Stability



Instability

Internal Stability of LTI Systems

Theorem (Internal Stability). The equation $\dot{x}(t) = Ax(t)$ is

1. Lyapunov stable if and only if all the eigenvalues of A have zero or negative real parts, and those with zero real part are associated with a Jordan block of order 1.
2. Asymptotically stable if and only if all eigenvalues of A have negative real parts.

Example. Consider

$$\dot{x}(t) = \begin{bmatrix} \boxed{0} & 0 & 0 \\ 0 & \boxed{0} & 0 \\ 0 & 0 & -1 \end{bmatrix} x(t).$$

The matrix A has eigenvalues $\lambda_0 = 0$ with multiplicity 2, and $\lambda_1 = -1$ with multiplicity 1. The eigenvalue $\lambda_1 = 0$ is associated to Jordan blocks of order 1, so the equation is Lyapunov stable. $\square$

Internal Stability of LTI Systems

Example. Now consider the equation

$$\dot{x}(t) = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & -1 \end{bmatrix} x(t).$$

The matrix A has the same eigenvalues and same multiplicities of the previous example. Now, however, the repeated eigenvalue $\lambda_1 = 0$ is associated to a Jordan block of order 2, so the equation is unstable.

Internal Stability of LTI Systems

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The matrix A has the same eigenvalues and same multiplicities of the previous example. Now, however, the repeated eigenvalue $\lambda_1 = 0$ is associated to a Jordan block of order 2, so the equation is unstable.

Indeed, we know that the solution of this equation is given by

$$\begin{aligned} x(t) = \mathbf{exp}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} t\right) x(0) &= \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} \\ &= \begin{bmatrix} x_1(0) + t x_2(0) \\ x_2(0) \\ e^{-t} x_3(0) \end{bmatrix}, \end{aligned}$$

from which we see that $x_1(t)$ grows unbounded if $x_2(0) \neq 0$. $\square$

Internal Stability of LTI Systems

A block diagram interpretation of the difference between the two systems

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x(t) \qquad \text{and} \qquad \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t)$$

Internal Stability of LTI Systems

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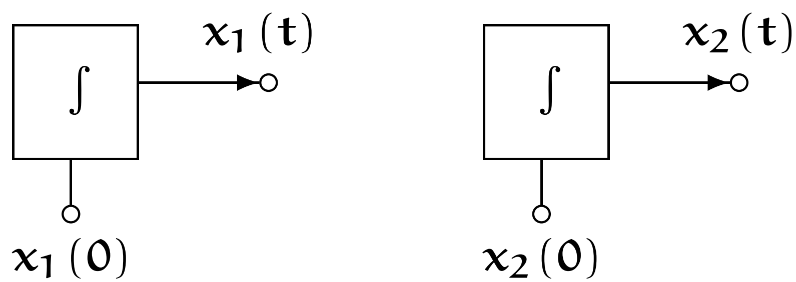
- the system on the left represents two **decoupled** integrators



Internal Stability of LTI Systems

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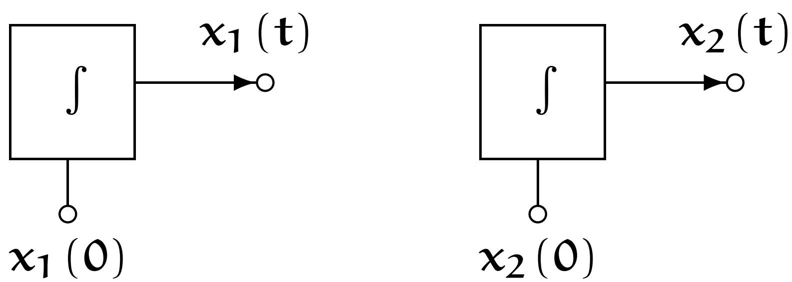
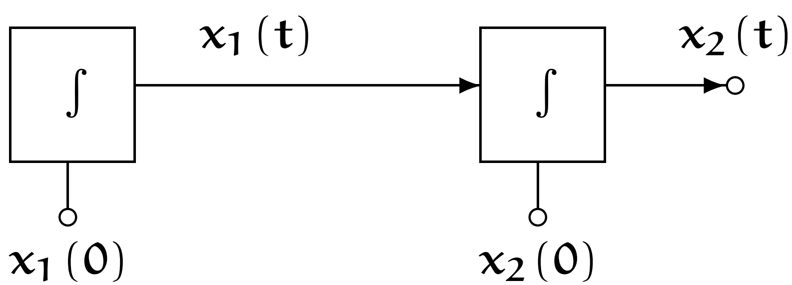


- the system on the right represents two **coupled** integrators



Internal Stability of LTI Systems

As discussed earlier, every pole of the transfer matrix

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

is an eigenvalue of $\mathbf{A}$. Thus,

$$\boxed{\textbf{asymptotic stability}} \Rightarrow \boxed{\textbf{BIBO stability}}$$

However, not all eigenvalues of $\mathbf{A}$ are necessarily poles of $\mathbf{G}(s)$, hence

$$\boxed{\textbf{asymptotic stability}} \not\Leftarrow \boxed{\textbf{BIBO stability}}$$

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Discrete-Time Systems

The concepts of BIBO stability and internal stability carry over to discrete-time systems. We summarise the main results.

Theorem (Discrete-time BIBO stability and impulse response). A discrete-time system with impulse response matrix $g[k] = [g_{ij}[k]]$ is BIBO stable if and only if every entry $g_{ij}[k]$ is absolutely summable, i.e., if

$$\sum_{k=0}^{\infty} |g[k]| \leq M \quad \text{for some finite constant } M \geq 0.$$

Theorem (Discrete-Time BIBO stability). A LTI discrete-time system with proper rational transfer function $G(z) = [G_{ij}(z)]$ is BIBO stable if and only if every pole of every entry $G_{ij}(z)$ has magnitude less than 1.

Discrete-Time Systems

Theorem (discrete-time internal stability). The system

$$x[k+1] = Ax[k]$$

is

1. Lyapunov stable if and only if all the eigenvalues of A have magnitude no greater than 1, and those eigenvalues with magnitude equal to 1 are associated to Jordan block of order 1.
2. Asymptotically stable if and only if all the eigenvalues of A have magnitude less than 1.

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Summary

- A system is BIBO stable if every bounded input excites a bounded output.
- BIBO stability is associated with the response of the system with zero initial state. A transfer matrix $\mathbf{G}(s)$ is BIBO stable iff all its poles have negative real part.
- Asymptotic stability is associated with the response of the system with zero input. A system $\dot{x} = Ax$ is asymptotically stable iff all the eigenvalues of A have negative real part.

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- BIBO stability is associated with the response of the system with zero initial state. A transfer matrix $\mathbf{G}(s)$ is BIBO stable iff all its poles have negative real part.
- Asymptotic stability is associated with the response of the system with zero input. A system $\dot{x} = Ax$ is asymptotically stable iff all the eigenvalues of A have negative real part.
- A system $\dot{x} = Ax$ is marginally stable, or Lyapunov stable, iff all the eigenvalues of A have nonpositive real part, and those eigenvalues with zero real part are "decoupled" (associated with Jordan blocks of order 1).