ELEC4410

Control Systems Design

Lecture 22: Introduction to Optimal Control and Estimation

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Outline

- Introduction
- The basic optimal control problem
- Optimal linear quadratic state feedback
- Optimal linear quadratic state estimation

Introduction

The state feedback and observer design approach is a fundamental tool in the control of state equation systems. However, it is not always the most useful method. Three obvious difficulties are:

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- The eigenvalues of the observer should be chosen faster than those of the closed-loop system. Is there any other criterion available to help decide one configuration over another?

The methods that we will now introduce give answers to these questions. We will see how the state feedback and observer gains can be found in an **optimal** way.

Some References for Further Reading

References

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(2) J.S. Bay. *Fundamentals of Linear State Space Systems*. McGraw-Hill, 1999.

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(4) B.D.O. Anderson and J.B. Moore. *Optimal Control. Linear quadratic methods*. Prentice-Hall International, 1989.

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The Basic Optimal Control Problem

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Conversely, a very precise and elegant system could be rejected as non-optimal because it is too expensive or it is too heavy or would take too long to develop.

The Basic Optimal Control Problem

The mathematical statement of the optimal control problem consists of

1. a description of the system to be controlled
2. a description of the system constraints and possible alternatives
3. a description of the task to be accomplished
4. a statement of the criterion for judging optimal performance

The Basic Optimal Control Problem

The dynamic system to be controlled is described in state variable form, i.e., in continuous-time by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

or in discrete-time by

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k], \qquad x[0] = x_0 \\ y[k] &= Cx[k] \end{aligned}$$

In the following, we assume that all the states are available as measurements, or otherwise, that the system is **observable**, so that an observer can be constructed to estimate the state.

The Basic Optimal Control Problem

System constraints will sometimes exist on allowable values of the state variables, or control inputs.

For example, the set of **admissible controls** could be the set of piecewise continuous vectors $\mathbf{u}(t) \in \mathcal{U}$ such that

$$\mathcal{U} = \{\mathbf{u}(t) : \|\mathbf{u}(t)\| < M \quad \text{for all } t.\}$$

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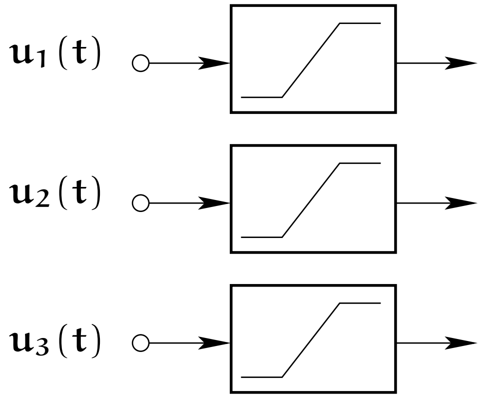
This constraint is very common in practice, and can represent **saturation in actuators**.

$$\left\| \begin{matrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{matrix} \right\|^2 = |u_1(t)|^2 + |u_2(t)|^2 + |u_3(t)|^2$$
$$< M^2, \quad \forall t$$



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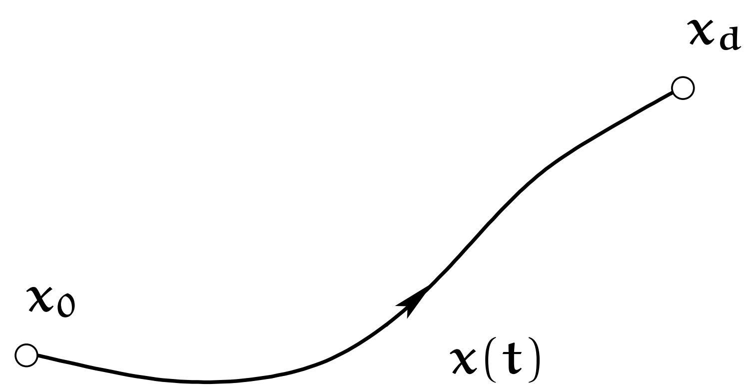

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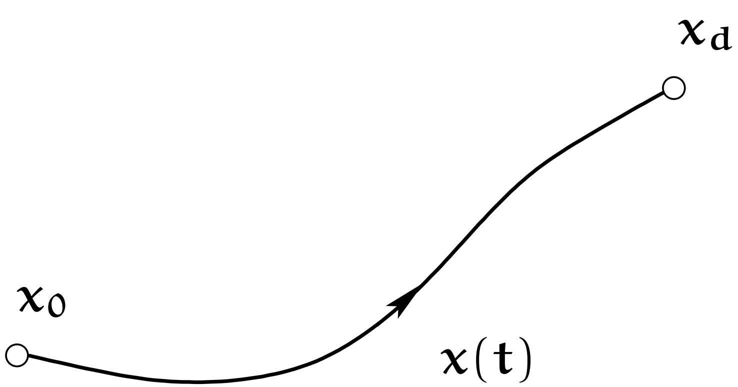

Often, the task to be performed is implicitly accounted for by the performance criterion.

The Basic Optimal Control Problem

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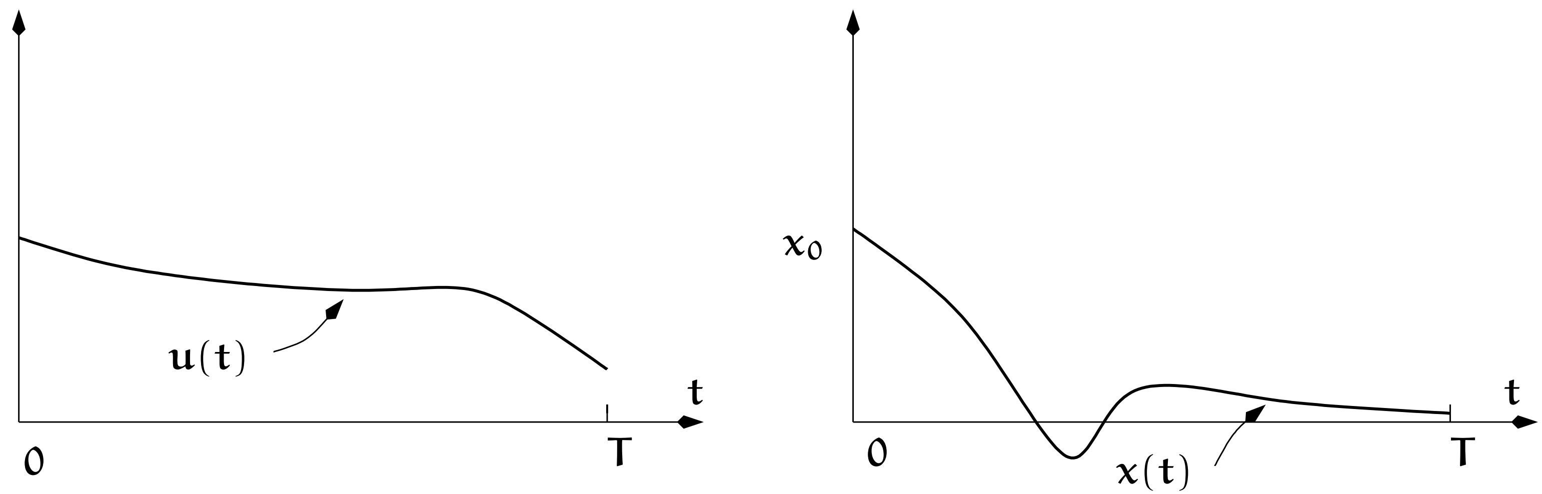

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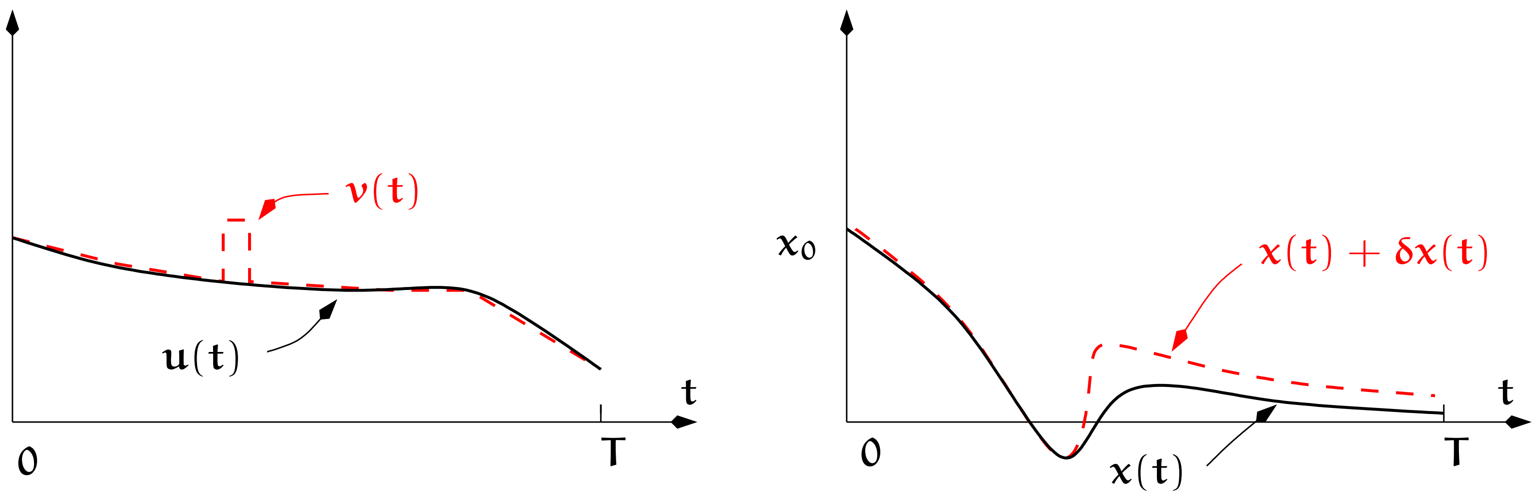

The input $u(t)$ generates the trajectory $x(t)$. A *variation* $v(t)$ in $u(t)$ generates a *different* trajectory $x(t) + \delta x(t)$.

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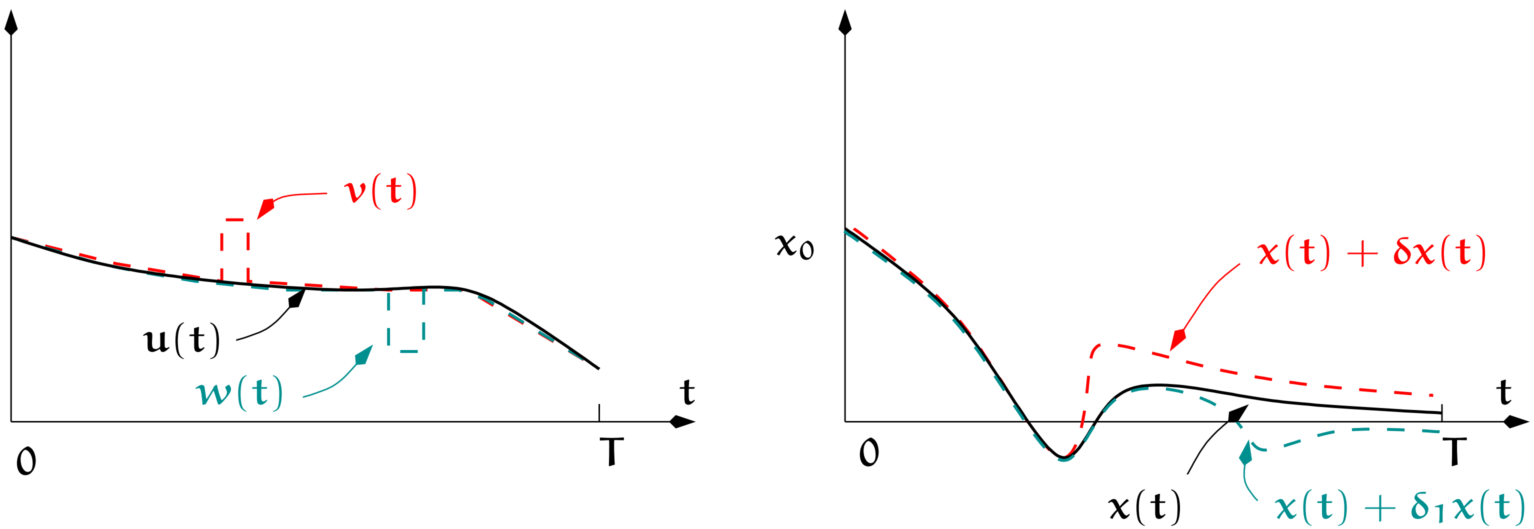

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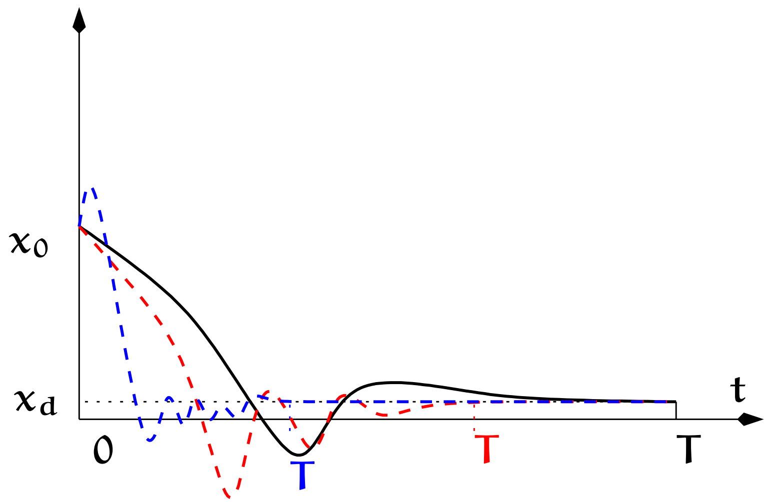
The Basic Optimal Control Problem

A common **performance criterion** is that of minimum time, in which we search for the control $u(t)$ which produces the fastest trajectory to get to the final desired state.



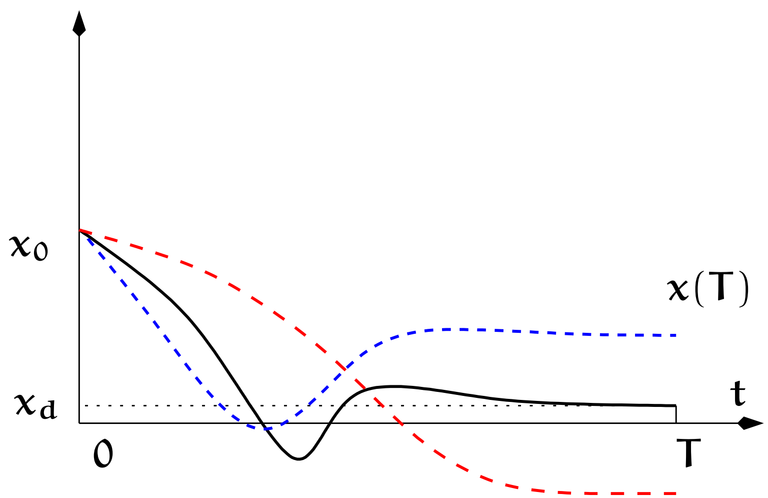
In this case the performance criterion to minimise can be simply expressed mathematically as

$$J = T$$

The Basic Optimal Control Problem

Another performance criterion could be the final error in achieving the desired final state in a prespecified time T, e.g.,

$$J = \|x(T)\|^2$$



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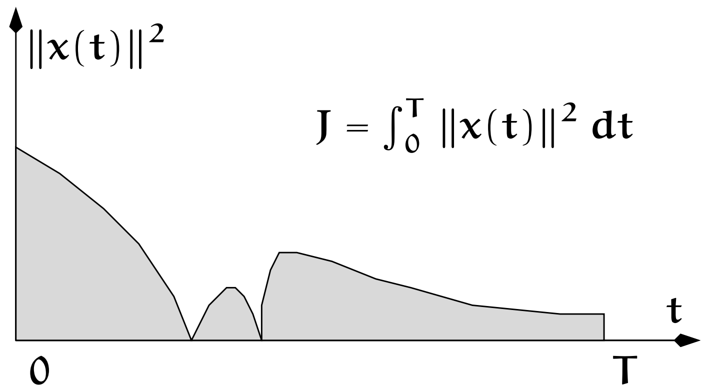
Another performance criterion could be to minimise the area under $\|x(t)\|^2$, as a way to select those controls that produce overall *small* transients in the generated trajectory between x_0 and the final state.



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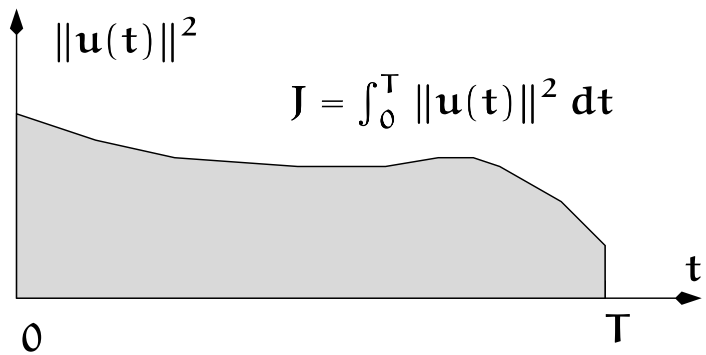

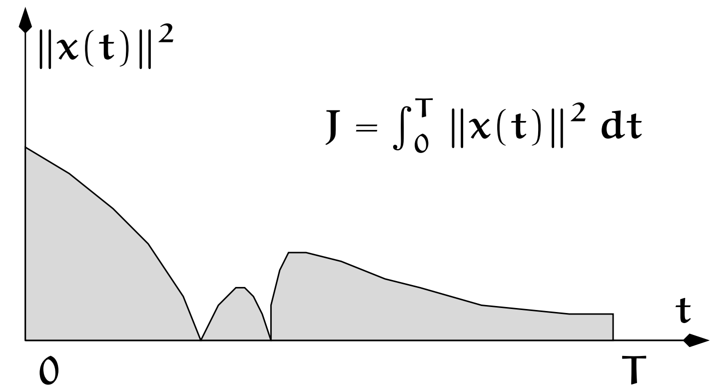

Yet another possible performance criterion could be to minimise the area under $\|u(t)\|^2$, as a way to select those controls that use the least *control effort*.

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Optimal Quadratic Control

A very important performance criterion which combines previous examples is the **quadratic performance criterion**. This criterion can be expressed in a general form as

$$\boxed{J = x^T(T)Sx(T) + \int_0^T \left[x^T(t)Qx(t) + u^T(t)Rx(t)\right]\, dt}$$

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The weighting matrices $\mathbf{S}$, $\mathbf{Q}$ and $\mathbf{R}$ allow a weighed tradeoff between the previous criteria. In particular, for example,

$$\begin{aligned}
\mathbf{S} = \mathbf{I}, \mathbf{Q} = 0, \mathbf{R} \to 0 \quad & \Rightarrow \mathbf{J} = \|\mathbf{x}(T)\|^2 \\
\mathbf{S} = 0, \mathbf{Q} = 0, \mathbf{R} = \mathbf{I} \quad & \Rightarrow \mathbf{J} = \int_0^T \|\mathbf{u}(t)\|^2\, dt
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$$S = 0, Q = 0, R = I \quad \Rightarrow J = \int_0^T \|u(t)\|^2\, dt$$

The matrices S and Q are symmetric and non negative definite, while R is symmetric and positive definite.

Interlude: Positive Definite Matrices

Recall an $n \times n$ symmetric matrix M is **positive definite** if

$$x^T M x > 0 \quad \text{for all } x \neq 0, x \in \mathbb{R}^n$$

and **non negative definite** if

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A symmetric matrix is positive definite (non negative definite) if and only if all its eigenvalues are positive (non negative).

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Example.

$$M_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \text{ is positive definite,} \quad \begin{bmatrix} x_1 & x_2 \end{bmatrix} M_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + x_2^2$$

$$M_2 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \text{ is non negative definite,} \quad \begin{bmatrix} x_1 & x_2 \end{bmatrix} M_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2$$

$$M_3 = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \text{ is not sign definite,} \quad \begin{bmatrix} x_1 & x_2 \end{bmatrix} M_3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 - x_2^2$$

Optimal Quadratic Control

The **quadratic performance criterion** for discrete-time systems is

$$\boxed{J_{0,N} = x_N^T S x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R x_k}$$

where for notational simplicity we wrote x_k to represent $x[k]$.

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When the final time N (the optimisation horizon) is set to $N = \infty$, we obtain an **infinite horizon** optimal control problem. In this case, for **stability**, we will require that $\lim_{N\to\infty} X_N = 0$,

$$J_{0,\infty} = \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R x_k$$

The continuous-time version of the infinite horizon criterion is

$$J_\infty = \int_0^\infty \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] \, dt$$

Optimal LQ State Feedback (LQR)

Theorem (LQR). Consider the state space system

$$\begin{aligned} \dot{x} &= Ax + Bu, & x \in \mathbb{R}^n, u \in \mathbb{R}^p \\ y &= Cx, & y \in \mathbb{R}^q \end{aligned}$$

and the performance criterion

$$J = \int_0^\infty \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] dt, \tag{J}$$

where Q is non negative definite and R is positive definite. Then the optimal control minimising (J) is given by the **linear** state feedback law

$$\boxed{u(t) = -Kx(t)} \quad \text{with} \quad \boxed{K = R^{-1} B^T P}$$

and where P is the unique positive definite solution to the matrix **Algebraic Riccati Equation** (ARE)

$$\boxed{A^T P + PA - PBR^{-1}B^T P + Q = 0}$$

Optimal LQ State Feedback (LQR)

Thus, to design an optimal state feedback law $\mathbf{u} = -\mathbf{K}\mathbf{x}$ minimising the cost

$$J = \int_0^\infty \left[\mathbf{x}^T(\tau)\mathbf{Q}\mathbf{x}(\tau) + \mathbf{u}^T(\tau)\mathbf{R}\mathbf{u}(\tau)\right] \, d\tau$$

we have to

1. Find the symmetric and positive definite solution of the algebraic Riccati equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = 0$$

2. Set $\boxed{\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}}$.

In MATLAB $\mathbf{K}$ and $\mathbf{P}$ can be computed using

```
[K,P] = lqr(A,B,Q,R);
```

Optimal LQ State Feedback (LQR)

The matrices $\mathbf{Q} \in \mathbb{R}^{n \times n}$ (non-negative definite) and $\mathbf{R} \in \mathbb{R}^{p \times p}$ (positive definite), are the **tuning parameters** of the problem.

- For example, the choice $\mathbf{Q} = \mathbf{C}^T\mathbf{C}$ and $\mathbf{R} = \lambda\mathbf{I}$, with $\lambda > 0$ corresponds to making a tradeoff between plant output and input "energies", with the cost

$$J = \int_0^{\infty} [\|y(\tau)\|^2 + \lambda\|u(\tau)\|^2]\, d\tau$$

  - λ small $\Rightarrow$ faster convergence of $y(t) \to 0$ but large control commands $u(t)$ (high gain control)
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It turns out that under some reasonable assumptions, the matrix $\mathbf{P}$ that solves the algebraic Riccati Equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = 0$$

exists. Furthermore, the corresponding closed-loop system is stable (i.e. $\mathbf{A} - \mathbf{B}\mathbf{K}$ has all its eigenvalues in the left half plane).

Optimal LQ State Feedback (LQR)

Example (LQR design). This example is from “Linear Optimal Control” by B.D.O. Anderson and J.B. Moore (Prentice-Hall, 1971). Suppose we have a transfer function

$$G(s) = \frac{1}{s(s+1)}$$

Writing $x_2 = \frac{1}{s+1}u, \quad x_1 = \frac{1}{s}x_2$ and $y = x_1$ we obtain the state space representation

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

Suppose the state feedback cost function is

$$J = \int_0^\infty \left[u^2 + x_1^2 + x_2^2\right] dt$$

This gives weighting matrices $Q = I$ and $R = 1$.

Optimal LQ State Feedback (LQR)

Example (Continuation). The algebraic Riccati equation requires $\mathbf{P}$ to satisfy

$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \mathbf{P} + \mathbf{P} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \mathbf{P} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{P} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

Since we require $\mathbf{P}$ to be positive definite and symmetric, set

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

This yields the simultaneous equations

$$\begin{aligned} p_{12}^2 - 1 &= 0 \\ 2(p_{12} - p_{22}) - p_{22}^2 + 1 &= 0 \\ p_{11} &= p_{12} + p_{12}p_{22} \end{aligned}$$

Optimal LQ State Feedback (LQR)

Example (Continuation). There are three possible solutions to the previous simultaneous equations, namely,

$$\mathbf{P} = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \quad \text{or} \quad \mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Only the last of these is positive definite, so it is the solution we require. This gives

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{P} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

with closed-loop poles at the eigenvalues of

$$\begin{aligned}(\mathbf{A} - \mathbf{B}\mathbf{K}) &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\end{aligned}$$

So the poles are the roots of the characteristic equation $s^2 + 2s + 1 = 0$ giving two poles each at -1. $\square$

Derivation of the Optimal LQR

We show the derivation of the LQR control result. We want to find a control law $\mathbf{u}$ to minimise the infinite horizon performance criterion

$$J = \int_0^{\infty} \left[\mathbf{x}^T(t) \mathbf{Q} \mathbf{x}(t) + \mathbf{u}^T(t) \mathbf{R} \mathbf{u}(t) \right] dt$$

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Suppose that $\mathbf{P}$ is the symmetric and positive definite solution of the Algebraic Riccati Equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = 0.$$

Derivation of the Optimal LQR

We show the derivation of the LQR control result. We want to find a control law $\mathbf{u}$ to minimise the infinite horizon performance criterion

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Define the **quadratic form** (function of t) $\mathbf{V}(t) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)$. We note that

$$\begin{aligned}
\dot{\mathbf{V}} &= \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{x} \\
&= (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \\
&= \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x} + \mathbf{u}^T\mathbf{B}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{u}
\end{aligned}$$

Derivation of the Optimal LQR

From the Algebraic Riccati Equation, we have that

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} + \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

so

$$\dot{\mathbf{V}} = -\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{x}^T(\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P})\mathbf{x} + \mathbf{u}^T\mathbf{B}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{u}$$

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Derivation of the Optimal LQR

From the Algebraic Riccati Equation, we have that

$$A^T P + PA = -Q + PBR^{-1}B^T P$$

so

$$\begin{aligned}\dot{V} &= -x^T Q x + x^T (PBR^{-1}B^T P)x + u^T B^T P x + x^T PBu + u^T R u - u^T R u \\ &= -\left[x^T Q x + u^T R u\right] + (B^T P x + R u)^T R^{-1} (B^T P x + R u)\end{aligned}$$

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Thus

$$\int_0^\infty \dot{V}(t)dt = -J + \int_0^\infty (B^T Px + Ru)^T R^{-1}(B^T Px + Ru)\, dt$$

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$$\begin{aligned}
& \int_0^\infty \dot{V}(t)\,dt = -J + \int_0^\infty (B^T P x + Ru)^T R^{-1} (B^T P x + Ru)\,dt \\
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$$\Leftrightarrow \quad J = x^T(0)Px(0) + \int_0^\infty (B^T Px + Ru)^T R^{-1}(B^T Px + Ru)\,dt$$

Derivation of the Optimal LQR

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Derivation of the Optimal LQR

We have arrived at

$$J = x^T(0)Px(0) + \int_0^\infty (B^TPx + Ru)^T R^{-1} (B^TPx + Ru)\, dt$$

Because the second term on the RHS is nonnegative, the minimum of J is clearly achieved when

$$\boxed{u = -R^{-1}B^TPx} = -Kx$$

and the minimum value of the cost is therefore

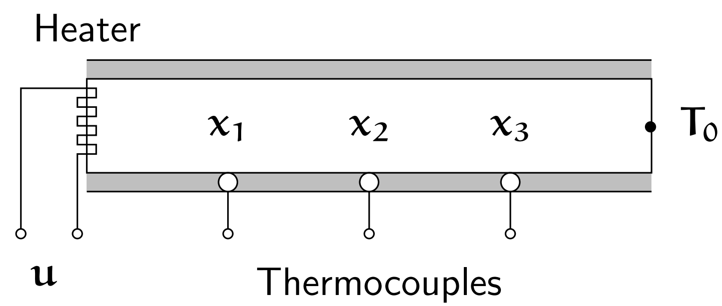
$$\boxed{\min_u J = x^T(0)Px(0).}$$

$\square$

LQR Matlab Example

Example (Furnace Control).

Consider the model of the Furnace seen in tutorial 10. The objective was to design a controller to achieve robust reference tracking in the thermocouple output x_2.



Let's design the feedback gains using the function `lqr`. We choose (for the **augmented** system (A_a, B_a) for integral action)

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad R = \lambda^{-k}, \quad \text{with } k = 0, 1, 2, 3$$

This choice of weights will represent the minimisation of the cost

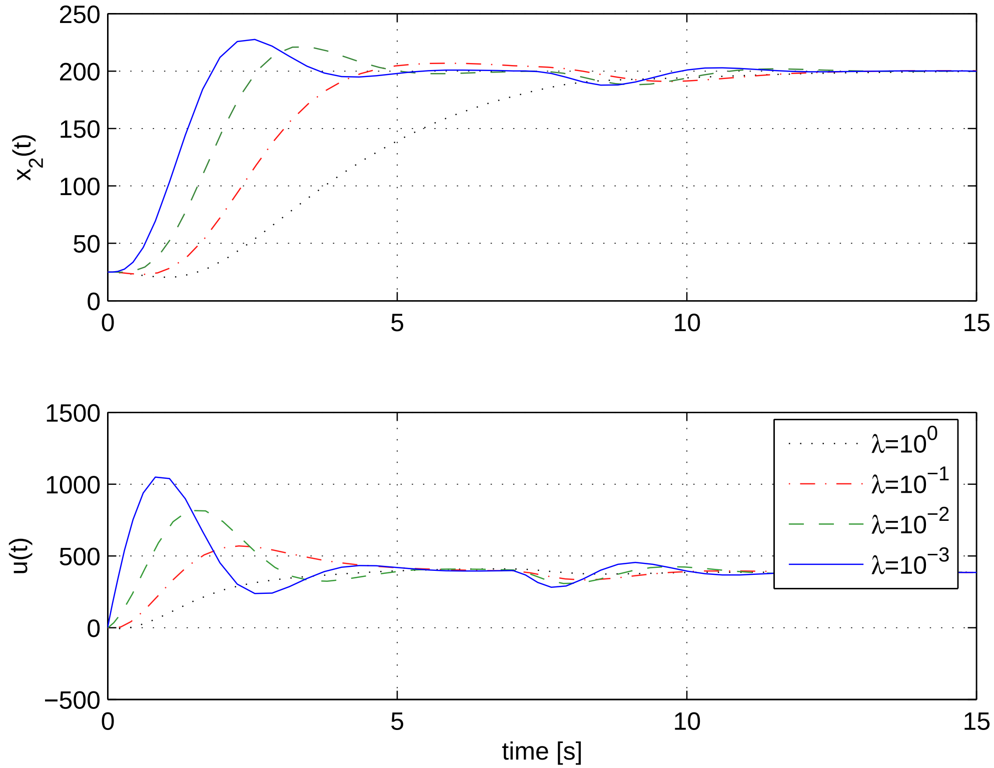
$$J = \int_0^\infty \left[\sigma^2(\tau) + \lambda u^2(\tau)\right] d\tau, \quad \text{where } \sigma \text{ is the integral of the tracking error.}$$

LQR Matlab Example

Example (Continuation. . .). We designed the gains using the function

```
Ka = lqr(Aa,Ba,Q,R);
```

for the various values of λ, and obtained a set of four gains.



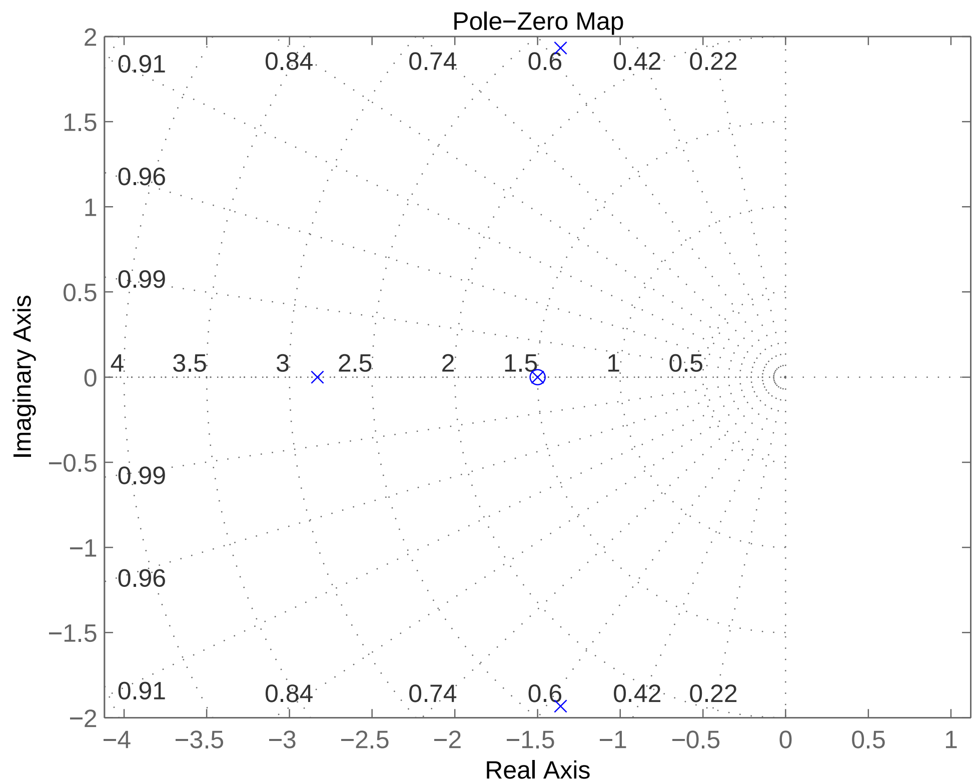
We simulated the response of the closed-loop system for each of them.

We can see that the smaller λ, the better the performance of $x_2(t)$, but the higher the control effort required.

LQR Matlab Example

Example (Continuation. . .). It is interesting to look at the closed-loop pole pattern achieved by the high-performance optimal controller ($\lambda = 10^{-3}$).

Note that one pole is placed on top of the slow stable open loop zero of the system (which prevents excessive overshoot) and the other 3 are distributed in a Butterworth-like configuration. $\square$



Discrete Optimal LQ State Feedback

The **quadratic performance criterion** for discrete-time systems is

$$\boxed{J_{0,N} = x_N^T S x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R x_k}$$

where for notational simplicity we wrote x_k to represent $x[k]$.

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When the final time N (the optimisation horizon) is set to $N = \infty$, we obtain an **infinite horizon** optimal control problem. In this case, for **stability**, we will require that $\lim_{N\to\infty} x_N = 0$,

$$J_{0,\infty} = \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R x_k$$

For discrete-time systems there is a parallel result to the continuous time LQR. The optimal control is also found via state feedback, but we unfortunately we need to solve a different Riccati equation.

Discrete Optimal LQ State Feedback

Theorem (Discrete-time LQR). Let

$$J = \sum_{k=0}^{\infty} \left[x_k^T Q x_k + u_k^T R u_k \right]$$

Then the optimal control is given by the state feedback law

$$u_k = -K x_k$$

with

$$K = (R + B^T P B)^{-1} B^T P A$$

and where P is the solution to the **discrete algebraic Riccati equation** (DARE)

$$\boxed{A^T P A - P - A^T P B (R + B^T P B)^{-1} B^T P A + Q = 0}$$

Discrete Optimal LQ State Feedback

Just as for the continuous case, under some reasonable assumptions there is a unique positive definite solution $\mathbf{P}$. Furthermore the corresponding closed-loop system is stable (i.e. $\mathbf{A} - \mathbf{BK}$ has all its eigenvalues in the unit circle).

In MATLAB $\mathbf{K}$ and $\mathbf{P}$ can be computed using

```
[K,P] = dlqr(A,B,Q,R);
```

Choosing

$$\mathbf{Q} = \mathbf{C}^T\mathbf{C} \quad \text{and} \quad \mathbf{R} = \lambda\mathbf{I}$$

gives

$$J = \sum_{t=0}^{\infty} \left[\|y_k\|^2 + \lambda \|u_k\|^2 \right]$$

As before, λ can then be used as a simple tuning parameter to trade off output performance against control action.

Outline

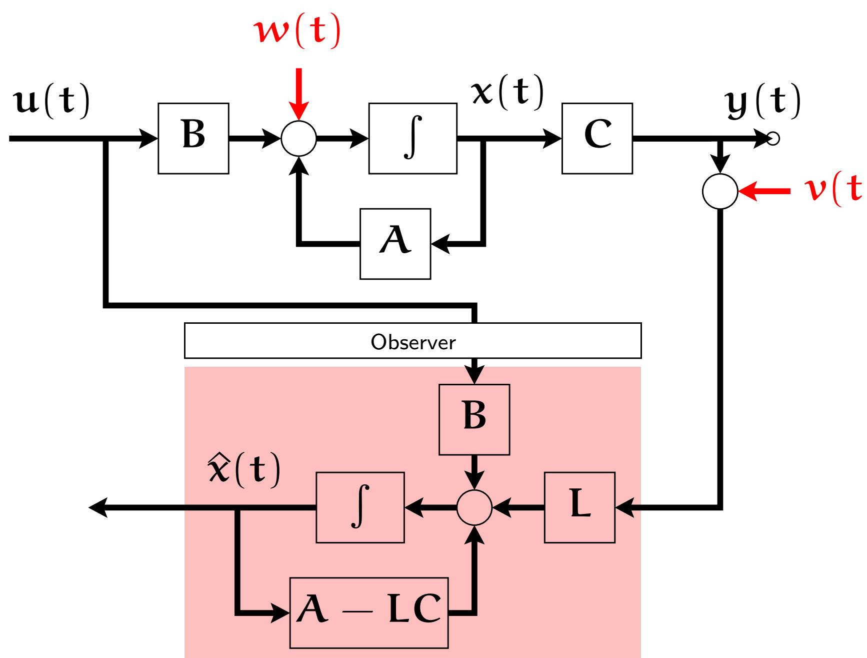
- Introduction
- The basic optimal control problem
- Optimal linear quadratic state feedback
- Optimal linear quadratic state estimation

Optimal State Estimation (LQE)

We now turn to optimal linear quadratic observers. The optimal LQ observer problem is **dual** to the LQ state feedback problem. However, optimal LQ observers have a **stochastic** interpretation, in that they are optimal in estimating the state in the presence of Gaussian noises corrupting the output measurements and the state.

Suppose we introduce state and output noise processes w and v so that

$$\begin{aligned} \dot{x} &= Ax + Bu + w \\ y &= Cx + v \end{aligned}$$



The signals w and v are **zero-mean stochastic Gaussian processes** uncorrelated in time and with each other. They have the following covariances:

$$E\left(ww^T\right) = W \quad \text{and} \quad E\left(vv^T\right) = V$$

Optimal State Estimation (LQE)

We can design an optimal LQ observer

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}})$$

with $\mathbf{L}$ given by

$$\mathbf{L} = \mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{V}^{-1}$$

where $\mathbf{P}$ is the solution to the algebraic Riccati equation

$$\boxed{\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathsf{T}} - \mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{C}\mathbf{P} + \boldsymbol{W} = 0}$$

It is usual to treat $\boldsymbol{W}$ and $\mathbf{V}$ as design parameters. For example it is common to assign $\boldsymbol{W} = \mathbf{B}\mathbf{B}^{\mathsf{T}}$ (so that effectively $\boldsymbol{w}$ is an input noise signal) and $\mathbf{V} = \mu\mathbf{I}$.

Optimal State Estimation (LQE)

High relative values of W will lead to large $\mathbf{L}$, so more weight is given to the output signal y, whereas high relative values of $\mathbf{V}$ will lead to small $\mathbf{L}$, so more weight is given to the input signal u. We may think of this as saying high values of $\mathbf{V}$ put more confidence in the model, giving **slower** observer feedback dynamics.

Such an optimal LQ state estimator is known as the (steady state) **Kalman filter**. In MATLAB, $\mathbf{L}$ and $\mathbf{P}$ can be computed as

```
[L,P] = lqr(A',C',W,V)';
```

Summary

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  - the system constraints
  - the task to be accomplished
  - a criterion to judge best performance

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$$J = x^T(T)Sx(T) + \int_0^T \left[x^T(t)Qx(t) + u^T(t)Rx(t) \right] dt$$

which is convenient to trade off different performance objectives of interest (such as minimum terminal state, minimum transients in the state, minimum control "effort", etc.)

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- The **Kalman filter** provides the best state estimates when the system is linear and corrupted with Gaussian noises with covariances W and $\mathbf{V}$. Then these matrices are used as the "weightings" in the performance criterion.
- The combination of an optimal LQR and LQE yield a **Linear Quadratic Gaussian (LQE)** controller.